

# REPRESENTATIONS FOR THE NON-GRADED VIRASORO-LIKE ALGEBRA\*

SHOULAN GAO AND CUIPO JIANG<sup>†</sup>

**ABSTRACT.** It is proved in this paper that an irreducible module over the non-graded Virasoro-like algebra  $\mathcal{L}$ , which satisfies a natural condition, is a *GHW* module or uniformly bounded. Furthermore, we give the complete classification of indecomposable  $\mathcal{L}$ -modules  $V = \bigoplus_{m,n \in \mathbb{Z}} \mathbb{C}v_{m,n}$  which satisfy  $L_{r,s}v_{m,n} \subseteq \mathbb{C}v_{r+m,s+n+1} + \mathbb{C}v_{r+m,s+n}$ .

## 1. Introduction

The Virasoro algebra  $Vir$  is the universal central extension of the Witt algebra with a basis  $\{L_i, c \mid i \in \mathbb{Z}\}$  such that for all  $i, j \in \mathbb{Z}$

$$[L_i, L_j] = (j - i)L_{i+j} + \frac{i^3 - i}{12}\delta_{i+j,0}c, \quad [L_i, c] = 0.$$

It plays an important role in many areas of mathematics and physics. Over the past decades many authors have studied the representation of  $Vir$  (see for example [1, 4, 5, 8]). One of the most important results proved by O. Mathieu is that an irreducible Harish-Chandra module over  $Vir$  is a highest weight module, a lowest weight module or a module of intermediate series ([9]). And it is known that a module of intermediate series over  $Vir$  is one of  $A_{a,b}, A(a'), B(a')$  or one of their quotients or submodules for suitable  $a, b \in \mathbb{C}, a' \in \mathbb{C} \cup \{\infty\}$ , where  $A_{a,b}, A(a'), B(a')$  all have a basis  $\{v_j \mid j \in \mathbb{Z}\}$ , such that  $c$  acts trivially and for all  $i, j \in \mathbb{Z}$

$$\begin{aligned} A_{a,b} : \quad & L_i v_j = (a + bi + j)v_{i+j}; \\ A(a') : \quad & L_i v_j = (i + j)v_{i+j}, \quad j \neq 0, \quad L_i v_0 = i[1 + (i + 1)a']v_i, \quad a' \in \mathbb{C}; \\ B(a') : \quad & L_i v_j = jv_{i+j}, \quad j \neq -i, \quad L_i v_{-i} = -i[1 + (i + 1)a']v_0, \quad a' \in \mathbb{C}; \\ A(\infty) : \quad & L_i v_j = (i + j)v_{i+j}, \quad j \neq 0, \quad L_i v_0 = i(i + 1)v_i; \\ B(\infty) : \quad & L_i v_j = jv_{i+j}, \quad j \neq -i, \quad L_i v_{-i} = -i(i + 1)v_0. \end{aligned}$$

Up to now many authors have studied various types of generalizations of the Virasoro algebra, such as the Virasoro-like algebra and its  $q$ -analog, generalized Witt algebras, the higher rank Virasoro algebra and the super-Virasoro algebra and so on, see for example ([2, 6, 3, 7, 10, 11, 12, 17]). However, all of these Lie algebras are graded. Due to the important applications in the theory of Hamiltonian operator and vertex operator algebras, infinite dimensional non-graded Lie algebras have been studied ([13, 14, 15, 16]). In [14], Su and Zhao calculated the second cohomology groups of Lie algebras of generalized Witt type and introduced the non-graded Virasoro-like Lie

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<sup>†</sup> Corresponding author: cpjiang@sjtu.edu.cn .

algebra  $W(\widetilde{\Gamma})$  ( $\Gamma$  is an additional subgroup of a field of characteristic zero) with a basis  $\{L'_{\alpha,i}, c \mid \alpha \in \Gamma, i \in \mathbb{Z}\}$  and the product

$$\begin{aligned} [L'_{\alpha,i}, L'_{\beta,j}] &= (\beta - \alpha)L'_{\alpha+\beta,i+j} + (j - i)L'_{\alpha+\beta,i+j-1} \\ &\quad + \frac{1}{12}\delta_{\alpha+\beta,0}[\delta_{i+j,-1}\alpha^3 + 3i\delta_{i+j,0}\alpha^2 + 3i(i-1)\delta_{i+j,1}\alpha + i(i-1)(i-2)\delta_{i+j,2}]c, \\ [L'_{\alpha,i}, c] &= 0. \end{aligned}$$

Since these Lie algebras are non-graded and have no Cartan subalgebras, we cannot define their weight modules as for graded Lie algebras. It makes the representation theory more difficult to study. So far not much has been achieved related to these non-graded Lie algebras. In this paper, we study representations of  $\mathcal{L} = W(\widetilde{\Gamma})$  over  $\mathbb{C}$  for  $\Gamma = \mathbb{Z}$ .

Let  $L_{\alpha,\beta} = L'_{\alpha,\beta+1}$ . Then the non-graded Virasoro-like Lie algebra  $\mathcal{L}$  is linearly spanned by  $\{L_{\alpha,\beta}, c \mid \alpha, \beta \in \mathbb{Z}\}$  with the following product

$$\begin{aligned} [L_{\alpha_1,\beta_1}, L_{\alpha_2,\beta_2}] &= (\alpha_2 - \alpha_1)L_{\alpha_1+\alpha_2,\beta_1+\beta_2+1} + (\beta_2 - \beta_1)L_{\alpha_1+\alpha_2,\beta_1+\beta_2} \\ &\quad + \frac{1}{12}\delta_{\alpha_1+\alpha_2,0}[\delta_{\beta_1+\beta_2,-3}\alpha_1^3 + 3\delta_{\beta_1+\beta_2,-2}(\beta_1+1)\alpha_1^2 \\ &\quad + 3\delta_{\beta_1+\beta_2,-1}\beta_1(\beta_1+1)\alpha_1 + \delta_{\beta_1+\beta_2,0}\beta_1(\beta_1^2-1)]c, \\ [L_{\alpha,\beta}, c] &= 0. \end{aligned}$$

Through out the paper, we study  $\mathcal{L}$ -modules  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  such that  $\dim V_{r,s} < \infty$  and  $L_{a,b}V_{r,s} \subseteq V_{r+a,s+b+2} + V_{r+a,s+b+1} + V_{r+a,s+b}$ . We prove in Section 2 that if  $V$  is irreducible, then  $V$  is either a *GHW* module or uniformly bounded. In sections 3-5, we study uniformly bounded modules of  $\mathcal{L}$  and give the complete classification of indecomposable  $\mathcal{L}$ -modules  $V = \bigoplus_{m,n \in \mathbb{Z}} V_{m,n}$  which satisfy the following two conditions: (1)  $\dim V_{m,n} \leq 1$ . (2)  $L_{\alpha,\beta}V_{m,n} \subseteq V_{\alpha+m,\beta+n+1} + V_{\alpha+m,\beta+n}$ .

## 2. GHW Modules

**Definition 2.1.** Let  $\alpha_1, \alpha_2 \in \mathbb{Z} \times \mathbb{Z}$ .  $\{\alpha_1, \alpha_2\}$  is called a  $\mathbb{Z}$ -basis of  $\mathbb{Z} \times \mathbb{Z}$ , if for each  $\alpha \in \mathbb{Z} \times \mathbb{Z}$ ,  $\alpha = k_1\alpha_1 + k_2\alpha_2$  for some  $k_1, k_2 \in \mathbb{Z}$ .

Set

$$e_1 = (1, 0), \quad e_2 = (0, 1).$$

In this paper, we always consider the  $\mathcal{L}$ -modules  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  such that  $\dim V_{r,s} < \infty$  and

$$L_{a,b}V_{r,s} \subseteq V_{r+a,s+b+2} + V_{r+a,s+b+1} + V_{r+a,s+b}. \quad (2.1)$$

**Definition 2.2.** Let  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  be a  $\mathcal{L}$ -module.  $V$  is called a *GHW* module if  $V$  is generated by a vector  $v$  and there is a  $\mathbb{Z}$ -basis  $\{\alpha_1, \alpha_2\}$  of  $\mathbb{Z} \times \mathbb{Z}$  such that

$$L_{\alpha}v = 0,$$

for all  $\alpha = k_1\alpha_1 + k_2\alpha_2 + k(1 - \delta_{k_1k_2,0})e_2$  with  $k_1, k_2 \in \mathbb{Z}_+$  and  $0 \leq k \leq k_1 + k_2$ .

**Lemma 2.3.** Let  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  be an irreducible  $\mathcal{L}$ -module and  $m, n \in \mathbb{Z}$  be such that  $|m| \geq 2$  and  $n \neq 0$ . Denote by  $\mathcal{S}_{m,n}$  the Lie subalgebra of  $\mathcal{L}$  generated by  $\{L_{m+i,n+j} \mid i = 1, 0, -1, j = 3, 0, -3\}$ . If there exists  $0 \neq v \in V$  such that  $\mathcal{S}_{m,n} \cdot v = 0$ , then  $V$  is a *GHW*  $\mathcal{L}$ -module.

**Proof: Case 1.**  $m, n > 0$ . Set

$$\alpha_1 = (nm - 1)e_1 + n^2e_2, \quad \alpha_2 = m^2e_1 + (mn + 1)e_2.$$

Then  $\alpha_1, \alpha_2$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z} \times \mathbb{Z}$ . Since

$$[L_{m,n}, L_{m-1,n}] = -L_{2m-1,2n+1},$$

$$[L_{m,n+3}, L_{m-1,n-3}] = -L_{2m-1,2n+1} - 6L_{2m-1,2n},$$

we have  $L_{2m-1,2n}, L_{2m-1,2n+1} \in \mathcal{S}_{m,n}$ . By the fact that

$$[L_{m,n}, L_{m-1,n-3}] = -L_{2m-1,2n-2} - 3L_{2m-1,2n-3},$$

$$[L_{m-1,n}, L_{m,n-3}] = L_{2m-1,2n-2} - 3L_{2m-1,2n-3},$$

we have  $L_{2m-1,2n-2}, L_{2m-1,2n-3} \in \mathcal{S}_{m,n}$ . Using the following Lie bracket relations

$$[L_{m,n}, L_{m-1,n+3}] = -L_{2m-1,2n+4} - 3L_{2m-1,2n+3},$$

$$[L_{m-1,n}, L_{m,n+3}] = L_{2m-1,2n+4} - 3L_{2m-1,2n+3},$$

we deduce that  $L_{2m-1,2n+4}, L_{2m-1,2n+3} \in \mathcal{S}_{m,n}$ . Therefore,

$$\{L_{2m-1,2n+k} | k = -3, -2, 0, 1, 3, 4\} \subseteq \mathcal{S}_{m,n}. \quad (2.2)$$

Similarly, we have

$$\{L_{2m,2n+k}, L_{2m+1,2n+k} | k = -3, -2, 0, 1, 3, 4\} \subseteq \mathcal{S}_{m,n}. \quad (2.3)$$

In general, for  $k \geq 3$ , we have

$$\{L_{km+i, kn+j} | i = -1, 0, 1; -3(k-1) \leq j \leq 4(k-1)\} \subseteq \mathcal{S}_{m,n}. \quad (2.4)$$

Since  $m \geq 2$ , by (2.2)-(2.3) and (2.4), considering  $k = n$  and  $k = m$  respectively, we have

$$\{L_{mn-1, n^2}, L_{mn, n^2}, L_{m^2+1, mn+1}, L_{m^2, mn+1}\} \subseteq \mathcal{S}_{m,n}.$$

Then it is easy to deduce that  $L_\alpha \subseteq \mathcal{S}_{m,n}$  for all  $\alpha = k_1\alpha_1 + k_2\alpha_2 + k(1 - \delta_{k_1k_2,0})e_2$  with  $k_1, k_2 \in \mathbb{Z}_+$  and  $0 \leq k \leq k_1 + k_2$ . By Definition 2.2, the lemma holds.

**Case 2.**  $m > 0, n < 0$ . Set

$$\alpha_1 = (1 - nm)e_1 - n^2e_2, \quad \alpha_2 = m^2e_1 + (1 + nm)e_2.$$

Then  $\alpha_1, \alpha_2$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z} \times \mathbb{Z}$ . Replacing  $k$  by  $-n$  and  $m$  in (2.2)-(2.4) respectively, we have

$$\{L_{-nm+1, -n^2}, L_{-nm, -n^2}, L_{m^2, mn+1}, L_{m^2-1, mn+1}\} \subseteq \mathcal{S}_{m,n}.$$

Therefore,  $L_\alpha \subseteq \mathcal{S}_{m,n}$  for all  $\alpha = k_1\alpha_1 + k_2\alpha_2 + k(1 - \delta_{k_1k_2,0})e_2$  with  $k_1, k_2 \in \mathbb{Z}_+$  and  $0 \leq k \leq k_1 + k_2$ .

For the case that  $m < 0, n > 0$ , set

$$\alpha_1 = (1 + nm)e_1 + n^2e_2, \quad \alpha_2 = -m^2e_1 + (1 - nm)e_2.$$

Then

$$\{L_{nm+1, n^2}, L_{nm, n^2}, L_{-m^2, -mn+1}, L_{-m^2-1, -mn+1}\} \subseteq \mathcal{S}_{m,n}.$$

For  $m < 0, n < 0$ , set

$$\alpha_1 = (-1 - nm)e_1 - n^2e_2, \quad \alpha_2 = -m^2e_1 + (1 - nm)e_2.$$

Then

$$\{L_{-nm-1, -n^2}, L_{-nm, -n^2}, L_{-m^2, -mn+1}, L_{-m^2+1, -mn+1}\} \subseteq \mathcal{S}_{m,n}.$$

We can similarly deduce the lemma for these two cases.  $\square$

**Lemma 2.4.** *Let  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  be an irreducible  $\mathcal{L}$ -module and  $m, n \in \mathbb{Z}$  be such that  $|m| \leq 1$  and  $|n| \geq 3$ . Denote by  $\mathcal{S}'_{m,n}$  the Lie subalgebra of  $\mathcal{L}$  generated by  $\{L_{m+i,n+j} | i = -1, 0, 1, j = -2, 1, 4\}$ . If there exists  $0 \neq v \in V$  such that  $\mathcal{S}'_{m,n} \cdot v = 0$ , then  $V$  is a GHW  $\mathcal{L}$ -module.*

**Proof:** We only consider the case that  $m = 1, n \geq 3$ . For other cases, the proof is similar. Set

$$\alpha_1 = (n-1)e_1 + n^2e_2, \quad \alpha_2 = e_1 + (n+1)e_2.$$

Then  $\alpha_1, \alpha_2$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z} \times \mathbb{Z}$ . Similar to the proof of Lemma 2.3, we have

$$\{L_{n-1,n^2}, L_{n,n^2}, L_{2,n+1}, L_{1,n+1}\} \subseteq \mathcal{S}'_{1,n}.$$

So  $L_\alpha \subseteq \mathcal{S}'_{m,n}$  for all  $\alpha = k_1\alpha_1 + k_2\alpha_2 + k(1 - \delta_{k_1k_2,0})e_2$  with  $k_1, k_2 \in \mathbb{Z}_+$  and  $0 \leq k \leq k_1 + k_2$ . Therefore, the lemma holds.

**Definition 2.5.** *Let  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  be a  $\mathcal{L}$ -module. If there exists a positive integer  $N$  such that  $\dim V_{r,s} \leq N$  for all  $r, s \in \mathbb{Z}$ , then  $V$  is called uniformly bounded.*

We now have the first main result of the paper.

**Theorem 2.6.** *Let  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  be an irreducible  $\mathcal{L}$ -module such that  $V$  is not uniformly bounded, then  $V$  is a GHW module.*

**Proof:** Suppose that  $V$  is not a GHW module. Let  $m, n \in \mathbb{Z}$ .

**Case 1.**  $|m| \geq 2, n \neq 0$ . Let  $S_{-m,-n} = \{L_{-m+i,-n+j} | i = 1, 0, -1, j = 3, 0, -3\}$  be the same as in Lemma 2.3. Then by Lemma 2.3, for any  $0 \neq v \in V_{m,n}$ , we have  $S_{-m,-n} \cdot v \neq \{0\}$ . By the assumption 2.1,

$$\left( \sum_{i=-1,0,1} \left( \sum_{j=-3,0,3} L_{-m+i,-n+j} \right) \right) |_{V_{m,n}} : V_{m,n} \longrightarrow \bigoplus_{i=-1,0,1} \left( \bigoplus_{j=-3}^5 V_{i,j} \right)$$

is an injection. Therefore,

$$\dim V_{m,n} \leq \sum_{i=-1,0,1} \left( \sum_{j=-3}^5 \dim V_{i,j} \right). \quad (2.5)$$

**Case 2.**  $|m| = 1, n \neq 0$ . Then by Lemma 2.4, for any  $0 \neq v \in V_{m,n}$ ,  $\mathcal{S}'_{-m,-n} \cdot v \neq \{0\}$ . Similar to the proof for case 1, we have

$$\dim V_{m,n} \leq \sum_{i=-1,0,1} \left( \sum_{j=-2}^6 \dim V_{i,j} \right). \quad (2.6)$$

**Case 3.**  $m = 0, n \neq 0$  or  $m \neq 0, n = 0$ . If  $m = 0, n \neq 0$ , by Lemma 2.4, for any  $0 \neq v \in V_{0,n}$ , we have  $\mathcal{S}'_{1,-n} \cdot v \neq \{0\}$ . Therefore,

$$\dim V_{0,n} \leq \sum_{i=0,1,2} \left( \sum_{j=-2}^6 \dim V_{i,j} \right). \quad (2.7)$$

If  $m \neq 0, n = 0$ , then  $n-1 = -1$  and similarly we have

$$\dim V_{m,0} \leq \sum_{i=-1,0,1} \left( \sum_{j=-4}^4 \dim V_{i,j} \right), \quad \text{if } |m| \geq 2, \quad (2.8)$$

$$\dim V_{m,0} \leq \sum_{i=-1,0,1} \left( \sum_{j=-3}^5 \dim V_{i,j} \right), \quad \text{if } |m| = 1. \quad (2.9)$$

It follows from (2.5)-(2.9) that  $V$  is uniformly bounded, a contradiction.  $\square$

### 3. Uniformly Bounded Modules

In this section, we discuss the  $\mathcal{L}$ -modules  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  such that  $\dim V_{r,s} < \infty$  and

$$L_{a,b}V_{r,s} \subseteq V_{r+a,s+b+1} + V_{r+a,s+b}. \quad (3.1)$$

**Lemma 3.1.** *Let  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  be a  $\mathcal{L}$ -module satisfying (3.1). Then  $c$  acts trivially on  $V$ .*

**Proof:** By the definition of  $\mathcal{L}$ , we have the following Lie bracket relation:

$$\begin{aligned} [L_{\alpha_1, \beta_1}, L_{\alpha_2, \beta_2}] &= (\alpha_2 - \alpha_1)L_{\alpha_1 + \alpha_2, \beta_1 + \beta_2 + 1} + (\beta_2 - \beta_1)L_{\alpha_1 + \alpha_2, \beta_1 + \beta_2} \\ &\quad + \frac{1}{12}\delta_{\alpha_1 + \alpha_2, 0}[\delta_{\beta_1 + \beta_2, -3}\alpha_1^3 + 3\delta_{\beta_1 + \beta_2, -2}(\beta_1 + 1)\alpha_1^2 \\ &\quad + 3\delta_{\beta_1 + \beta_2, -1}\beta_1(\beta_1 + 1)\alpha_1 + \delta_{\beta_1 + \beta_2, 0}\beta_1(\beta_1^2 - 1)]c. \end{aligned}$$

Let  $\alpha_1 = -\alpha_2$ ,  $\beta_1 + \beta_2 = k$ ,  $k = -3, -2, -1, 0$  respectively, then by (3.1) and the above relation, we have

$$c \cdot V_{m,n} \subseteq \bigcap_{k=-3}^0 \left( \bigoplus_{k \leq i \leq k+2} V_{m,n+i} \right) = \{0\},$$

for all  $m, n \in \mathbb{Z}$ . The lemma is proved.  $\square$

For  $m, n \in \mathbb{Z}$  and  $L_{m,n} \in \mathcal{L}$ , by the assumption (3.1), for each  $v \in V_{r,s}$ ,

$$L_{m,n} \cdot v = v_1 + v_2,$$

where  $v_1 \in V_{r+m,s+n+1}$ ,  $v_2 \in V_{r+m,s+n}$ ,  $r, s \in \mathbb{Z}$ . Define  $P_{m,n} : V_{r,s} \rightarrow V_{r+m,s+n+1}$  and  $Q_{m,n} : V_{r,s} \rightarrow V_{r+m,s+n}$  by

$$P_{m,n} \cdot v = v_1, \quad Q_{m,n} \cdot v = v_2.$$

Then

$$L_{m,n} = P_{m,n} + Q_{m,n}.$$

**Lemma 3.2.** *Let  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  be a  $\mathcal{L}$ -module satisfying (3.1). For  $h, k, r, s \in \mathbb{Z}$ , we have*

$$[P_{h,k}, P_{r,s}] = (r - h)P_{h+r,k+s+1}, \quad (3.2)$$

$$[P_{h,k}, Q_{r,s}] + [Q_{h,k}, P_{r,s}] = (r - h)Q_{h+r,k+s+1} + (s - k)P_{h+r,k+s}, \quad (3.3)$$

$$[Q_{h,k}, Q_{r,s}] = (s - k)Q_{h+r,k+s}. \quad (3.4)$$

In particular,

$$\mathcal{P}_{-1} = \text{span}\{P_{m,-1} \mid m \in \mathbb{Z}\}$$

and

$$\mathcal{Q}_0 = \text{span}\{Q_{0,n} \mid n \in \mathbb{Z}\}$$

are centerless Virasoro Lie algebras and for each  $m, n \in \mathbb{Z}$ ,

$$V_m^L = \bigoplus_{s \in \mathbb{Z}} V_{m,s}, \quad V_n^R = \bigoplus_{r \in \mathbb{Z}} V_{r,n}$$

are  $\mathcal{Q}_0$  and  $\mathcal{P}_{-1}$  modules respectively.

In the following sections of the paper, we discuss the indecomposable  $\mathcal{L}$ -modules  $V = \bigoplus_{r,s \in \mathbb{Z}} V_{r,s}$  satisfying (3.1) and  $\dim V_{m,n} \leq 1$ , for all  $m, n \in \mathbb{Z}$ . By Lemma 3.2, we can assume that

$$P_{0,-1}u = (\lambda + m)u, \quad Q_{0,0}v = (\mu + n)v, \quad (3.5)$$

for  $u \in V_{m,-1}$ ,  $v \in V_{0,n}$ , where  $\lambda, \mu \in \mathbb{C}$  are two fixed complex numbers.

For convenience, denote

$$P_{h,k}v_{m,n} = f_{h,k}(m, n)v_{h+m, n+k+1}, \quad Q_{h,k}v_{m,n} = g_{h,k}(m, n)v_{m+h, n+k},$$

for  $m, n, h, k \in \mathbb{Z}$ . Then

$$\begin{aligned} & f_{h,k}(m+r, n+s+1)f_{r,s}(m, n) - f_{r,s}(h+m, k+n+1)f_{h,k}(m, n) \\ &= (r-h)f_{r+h, s+k+1}(m, n) \end{aligned} \quad (3.6)$$

$$\begin{aligned} & f_{h,k}(r+m, s+n)g_{r,s}(m, n) + g_{h,k}(m+r, s+n+1)f_{r,s}(m, n) \\ & - f_{r,s}(h+m, k+n)g_{h,k}(m, n) - g_{r,s}(m+h, k+n+1)f_{h,k}(m, n) \\ &= (r-h)g_{h+r, k+s+1}(m, n) + (s-k)f_{h+r, k+s}(m, n), \end{aligned} \quad (3.7)$$

$$\begin{aligned} & g_{h,k}(m+r, s+n)g_{r,s}(m, n) - g_{r,s}(m+h, n+k)g_{h,k}(m, n) \\ &= (s-k)g_{h+r, k+s}(m, n). \end{aligned} \quad (3.8)$$

By (3.5), we have

$$f_{0,-1}(m, -1) = \lambda + m, \quad g_{0,0}(0, n) = \mu + n. \quad (3.9)$$

**Lemma 3.3.** *We have*

$$f_{0,-1}(m, n) = \lambda + m \quad (3.10)$$

$$g_{0,0}(m, n) = \mu + n, \quad (3.11)$$

for all  $m, n \in \mathbb{Z}$ .

**Proof:** Let  $m = r = s = k = 0$  in (3.8), then

$$g_{h,0}(0, n)g_{0,0}(0, n) = g_{0,0}(h, n)g_{h,0}(0, n), \quad (3.12)$$

for all  $n, h \in \mathbb{Z}$ . Let  $k = -1$ ,  $s = r = h = 0$  in (3.6), then

$$f_{0,-1}(m, n+1)f_{0,0}(m, n) = f_{0,0}(m, n)f_{0,-1}(m, n).$$

If for each  $h \in \mathbb{Z}$ , there exists  $n \in \mathbb{Z}$  such that  $g_{h,0}(0, n) \neq 0$ , then (3.11) follows from (3.12).

Suppose there exists  $h \in \mathbb{Z}$  such that  $g_{h,0}(0, n) = 0$  for all  $n \in \mathbb{Z}$ . Let  $m = r = k = 0$  and  $s \neq 0$  in (3.8), we have

$$g_{h,s}(0, n) = 0,$$

for all  $s, n \in \mathbb{Z}$ . Let  $r = -h$  and  $s = k = m = 0$  in (3.8), then

$$g_{h,0}(-h, n)g_{-h,0}(0, n) = 0,$$

for all  $n \in \mathbb{Z}$ . Let  $r = -h$  and  $k = m = 0$  in (3.8), we have

$$g_{h,0}(-h, s+n)g_{-h,s}(0, n) = sg_{0,s}(0, n).$$

So for  $s \neq 0$  and  $n \in \mathbb{Z}$ , if  $g_{h,0}(-h, s+n) = 0$ , then  $g_{0,s}(0, n) = 0$ . If  $g_{h,0}(-h, s+n) \neq 0$ , then  $g_{-h,0}(0, s+n) = 0$ . Since  $g_{h,-s}(0, s+n) = 0$ , let  $m = s = 0$ ,  $r = -h$ ,  $k = -s$  and replace  $n$  by  $n+s$  in (3.8), then we have

$$g_{0,-s}(0, s+n) = 0.$$

Therefore for  $s, n \in \mathbb{Z}$ ,  $s \neq 0$ , we have

$$g_{0,s}(0, n)g_{0,-s}(0, s + n) = 0.$$

Let  $m = h = r = 0$ ,  $k = -s$  in (3.8), then

$$g_{0,0}(0, n) = 0$$

for all  $n \in \mathbb{Z}$ , which is not true.  $\square$

We now have the second main result of the paper.

**Theorem 3.4.** *Let  $V = \bigoplus_{r,s \in \mathbb{Z}} \mathbb{C}v_{r,s}$  be an indecomposable  $\mathcal{L}$ -module satisfying (3.1) and (3.5). Then  $V$  satisfies one of the following situations:*

(1)  $A_{a,\lambda,\mu} : L_{r,s}v_{m,n} = (ar + \lambda + m)v_{m+r,n+s+1} + (as + \mu + n)v_{m+r,n+s}$ , where  $\lambda, \mu, a \in \mathbb{C}$ ;

(2)  $A_{0,\lambda,\mu} : L_{r,s}v_{m,n} = \frac{\mu + n}{\mu + n + s + 1}(\lambda + m)v_{m+r,n+s+1} + \frac{\lambda + m}{\lambda + m + r}(\mu + n)v_{m+r,n+s}$ ,

where  $\lambda, \mu \notin \mathbb{Z}$ ;

(3)  $A_{1,\lambda,\mu} : L_{r,s}v_{m,n} = \frac{\mu + n + s + 1}{\mu + n}(r + \lambda + m)v_{m+r,n+s+1} + \frac{\lambda + m + r}{\lambda + m}(s + \mu + n)v_{m+r,n+s}$ ,

where  $\lambda, \mu \notin \mathbb{Z}$ ;

(4)  $A_{1,0,\lambda,\mu} : L_{r,s}v_{m,n} = (r + \lambda + m)v_{m+r,n+s+1} + \frac{\lambda + m + r}{\lambda + m}(\mu + n)v_{m+r,n+s}$ , where  $\lambda \notin \mathbb{Z}$ ;

(5)  $B_{1,0,\lambda,\mu} : L_{r,s}v_{m,n} = \frac{\mu + n}{\mu + n + s + 1}(r + \lambda + m)v_{m+r,n+s+1} + (\mu + n)v_{m+r,n+s}$ , where  $\mu \notin \mathbb{Z}$ ;

(6)  $A_{0,1,\lambda,\mu} : L_{r,s}v_{m,n} = (\lambda + m)v_{m+r,n+s+1} + \frac{\lambda + m}{\lambda + m + r}(s + \mu + n)v_{m+r,n+s}$ , where  $\lambda \notin \mathbb{Z}$ ;

(7)  $B_{0,1,\lambda,\mu} : L_{r,s}v_{m,n} = \frac{\mu + n + s + 1}{\mu + n}(\lambda + m)v_{m+r,n+s+1} + (s + \mu + n)v_{m+r,n+s}$ , where  $\mu \notin \mathbb{Z}$ .

We will prove the theorem through lemmas in the next two sections. It is easy to check that (1)-(7) are indecomposable representations of  $\mathcal{L}$ .

#### 4. Proof of Theorem 3.4 for some cases

In this section we suppose that

$$Q_{0,s}v_{m,n} = (a_ms + \mu + n)v_{m,n+s}, \quad (4.1)$$

$$P_{r,-1}v_{m,n} = (b_nr + \lambda + m)v_{m+r,n}, \quad (4.2)$$

where  $a_m, b_n \in \mathbb{C}$ . In fact, if  $\mu \neq 0$  or  $\lambda \neq 0$ , then  $Q_{0,s}$  or  $P_{r,-1}$  has to be the form of (4.1) or (4.2), since  $V_m^L = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}v_{m,n}$  and  $V_n^R = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}v_{m,n}$  are  $\mathcal{Q}_0$  and  $\mathcal{P}_{-1}$  modules respectively.

**Lemma 4.1.**  *$a_m = a$  for all  $m \in \mathbb{Z}$ . That is,*

$$g_{0,s}(m, n) = as + \mu + n,$$

for all  $m \in \mathbb{Z}$ .

**Proof:** By the fact that  $[L_{0,-1}, L_{r,-1}] = rL_{r,-1}$  and (3.9), we have

$$g_{0,-1}(m + r, n)f_{r,-1}(m, n) = g_{0,-1}(m, n)f_{r,-1}(m, n - 1). \quad (4.3)$$

So

$$(\mu + n - a_{m+r})(b_n r + \lambda + m) = (\mu + n - a_m)(b_{n-1} r + \lambda + m), \quad (4.4)$$

for all  $m, n, r \in \mathbb{Z}$ . Replacing  $r$  by  $-r$  and  $m$  by  $m + r$  in (4.4), we have

$$(\mu + n - a_m)(-b_n r + \lambda + m + r) = (\mu + n - a_{m+r})(-b_{n-1} r + \lambda + m + r). \quad (4.5)$$

By (4.4) and (4.5) we have  $(b_n + b_{n-1} - 1)r(a_{m+r} - a_m) = 0$ . Then

$$b_n + b_{n-1} = 1 \text{ or } a_{m+r} = a_m,$$

for all  $n, m, r \in \mathbb{Z}$ . If  $b_n + b_{n-1} = 1$  for all  $n \in \mathbb{Z}$ , then

$$b_{2k} = b_0, \quad b_{2k+1} = 1 - b_0, \quad k \in \mathbb{Z}.$$

Let  $n = 2k$  and  $n = 2k + 1$  in (4.4) respectively, then we have

$$(2b_0 - 1)r(\mu + 2k - a_m) = (b_0 r + \lambda + m)(a_{m+r} - a_m), \quad (4.6)$$

$$(2b_0 - 1)r(\mu + 2k + 1 - a_{m+r}) = (b_0 r + \lambda + m)(a_m - a_{m+r}), \quad (4.7)$$

for all  $k, m, r \in \mathbb{Z}$ . By (4.6) and (4.7), we have

$$(2b_0 - 1)(2\mu + 4k - 1 - a_m - a_{m+r}) = 0,$$

for all  $k \in \mathbb{Z}$ . There always exists  $k \in \mathbb{Z}$  such that  $(2\mu + 4k - 1 - a_m - a_{m+r}) \neq 0$  for

all  $m, r \in \mathbb{Z}$ . Therefore,  $b_0 = \frac{1}{2}$ . Then  $b_n = \frac{1}{2}$  for all  $n \in \mathbb{Z}$ . By (4.4), we have

$$\left(\frac{1}{2}r + \lambda + m\right)(a_m - a_{m+r}) = 0, \quad (4.8)$$

for all  $m, r \in \mathbb{Z}$ . Let  $m = 0$  in (4.8), then

$$\left(\frac{1}{2}r + \lambda\right)(a_0 - a_r) = 0.$$

If  $\lambda \notin \frac{1}{2}\mathbb{Z}$ , then  $a_r = a_0$  for all  $r \in \mathbb{Z}$ . If  $\lambda \in \frac{1}{2}\mathbb{Z}$ , by (4.8), we have

$$a_r = a_0,$$

for all  $r \neq -2\lambda$ . Let  $m = -2\lambda, r \neq 0$  and  $r \neq 2\lambda$  in (4.8), then we have  $a_{-2\lambda} = a_0$ . So  $a_m = a$  for all  $m \in \mathbb{Z}$ .  $\square$

From the relation that  $[L_{r,-1}, L_{0,s}] = -rL_{r,s} + (s+1)L_{r,s-1}$ , we have

$$(b_{n+s+1}r + \lambda + m)f_{0,s}(m, n) - (b_n r + \lambda + m)f_{0,s}(m + r, n) = -rf_{r,s}(m, n), \quad (4.9)$$

$$\begin{aligned} & (b_{n+s+1} - b_n)r(as + \mu + n) + f_{0,s}(m, n)g_{r,-1}(m, n + s + 1) \\ & - g_{r,-1}(m, n)f_{0,s}(m + r, n - 1) = -rg_{r,s}(m, n) + (s+1)f_{r,s-1}(m, n), \end{aligned} \quad (4.10)$$

$$g_{r,-1}(m, n + s)(as + \mu + n) - g_{r,-1}(m, n)(as + \mu + n - 1) = (s+1)g_{r,s-1}(m, n). \quad (4.11)$$

By the fact that  $[L_{0,-1}, L_{r,-1}] = rL_{r,-1}$ , we have

$$g_{r,-1}(m, n)(\mu + n - 1 - a) = g_{r,-1}(m, n - 1)(\mu + n - a). \quad (4.12)$$

By (4.4), if  $\mu + n - a \neq 0$  for all  $n \in \mathbb{Z}$ , then  $b_n = b$  for all  $n \in \mathbb{Z}$ .

**Lemma 4.2.** Suppose that  $\mu + n - a \neq 0$  for all  $n \in \mathbb{Z}$ .

(1) If  $b \neq 0, 1$ , then  $a=b$  and

$$\begin{cases} f_{r,s}(m, n) = ar + \lambda + m, \\ g_{r,s}(m, n) = as + \mu + n; \end{cases}$$

(2) If  $b \in \{0, 1\}$ , then  $a \in \{0, 1\}$ .



**Proof:** By (4.9)-(4.10), we have

$$(br + \lambda + m)(f_{0,s}(m, n) - f_{0,s}(m + r, n)) = -rf_{r,s}(m, n), \quad (4.13)$$

$$\begin{aligned} f_{0,s}(m, n)g_{r,-1}(m, n + s + 1) - g_{r,-1}(m, n)f_{0,s}(m + r, n - 1) \\ = -rg_{r,s}(m, n) + (s + 1)f_{r,s-1}(m, n). \end{aligned} \quad (4.14)$$

By (4.12) and  $\mu + n - a \neq 0$  for all  $n \in \mathbb{Z}$ , we can assume that

$$\frac{g_{r,-1}(m, n)}{\mu + n - a} = \frac{g_{r,-1}(m, n - 1)}{\mu + n - 1 - a} = k_{r,m},$$

for all  $m, r, n \in \mathbb{Z}$ . Note that  $k_{0,m} = 1$  for all  $m \in \mathbb{Z}$ . By (4.11), we have

$$g_{r,s}(m, n) = k_{r,m}(as + \mu + n), \quad s \neq -2.$$

By (3.8), we have

$$k_{r,m}k_{-r,m+r} = 1, \quad (4.15)$$

$$k_{h,m+r}k_{r,m} = k_{r,m+h}k_{h,m} = k_{h+r,m}, \quad (4.16)$$

for all  $r, m, h \in \mathbb{Z}$ . From (4.15) we can see that  $k_{r,m} \neq 0$  for all  $r, m \in \mathbb{Z}$ . Let  $h = 0, k = -2, s = 0$  in (3.8), we have  $g_{r,-2}(m, n) = k_{r,m}(-2a + \mu + n)$ . Then

$$g_{r,s}(m, n) = k_{r,m}(as + \mu + n), \quad (4.17)$$

for all  $r, s, m, n \in \mathbb{Z}$ . In (3.7), let  $h = 0, r = 0, k = -1$ , then we have

$$f_{0,s}(m, n)(\mu + n + s + 1 - a) - (\mu + n - a)f_{0,s}(m, n - 1) = (s + 1)f_{0,s-1}(m, n). \quad (4.18)$$

So

$$f_{0,0}(m, n)(\mu + n + 1 - a) - f_{0,0}(m, n - 1)(\mu + n - a) = \lambda + m, \quad (4.19)$$

and

$$(\mu + n + 1 - a)f_{0,0}(m, n) = n(\lambda + m) + (\mu + 1 - a)f_{0,0}(m, 0). \quad (4.20)$$

Letting  $n = 1, s = 0$  in (4.14), we obtain

$$f_{0,0}(m, 1)k_{r,m}(\mu + 2 - a) - k_{r,m}(\mu + 1 - a)f_{0,0}(m + r, 0) = -rk_{r,m}(\mu + 1) + br + \lambda + m.$$

By (4.20), we have

$$[\lambda + m + f_{0,0}(m, 0)(\mu + 1 - a)]k_{r,m} - f_{0,0}(m + r, 0)(\mu + 1 - a)k_{r,m} = br + \lambda + m - rk_{r,m}(\mu + 1). \quad (4.21)$$

So

$$k_{r,m}(\mu + 1 - a)(f_{0,0}(m, 0) - f_{0,0}(m + r, 0)) + k_{r,m}(\lambda + m + r\mu + r) = br + \lambda + m. \quad (4.22)$$

Replace  $r$  by  $-r$  and  $m$  by  $m + r$  in (4.21) respectively, then

$$k_{-r,m+r}(\mu + 1 - a)(f_{0,0}(m + r, 0) - f_{0,0}(m, 0)) + k_{-r,m+r}(\lambda + m - r\mu) = -br + \lambda + m + r.$$

Then by (4.15), we have

$$k_{r,m}^2(-br + \lambda + m + r) - (2\lambda + 2m + r)k_{r,m} + br + \lambda + m = 0,$$

i.e.,

$$(k_{r,m} - 1)[(-br + \lambda + m + r)k_{r,m} - (br + \lambda + m)] = 0. \quad (4.23)$$

If  $b = \frac{1}{2}$ , then  $k_{r,m} = 1$  for all  $r, m \in \mathbb{Z}$ .

Suppose  $b \neq \frac{1}{2}$  and there exist  $r, m \in \mathbb{Z}$  such that  $-br + \lambda + m + r \neq 0$  and  $k_{r,m} = \frac{br + \lambda + m}{-br + \lambda + m + r} \neq 1$ . For  $h \neq -r, h \neq 0$ , if  $k_{h,m+r} = 1$ , then by (4.16) we have

$$k_{h+r,m} = k_{r,m} \neq 1.$$

Then

$$k_{h+r,m} = \frac{b(h+r) + \lambda + m}{-b(h+r) + \lambda + m + r + h} = \frac{br + \lambda + m}{-br + \lambda + m + r}.$$

This implies that  $b = \frac{1}{2}$ , a contradiction. So  $k_{h,m+r} \neq 1$  for  $h \neq 0, h \neq -r$ . Then by (4.16),

$$\frac{bh + \lambda + m + r}{-bh + \lambda + m + r + h} \cdot \frac{br + \lambda + m}{-br + \lambda + m + r} = \frac{b(h+r) + \lambda + m}{-b(h+r) + \lambda + m + r + h}.$$

This forces that  $b = 0$  or  $1$ . By the assumption that  $b \neq 0, 1$ , we have  $k_{r,m} = 1$ . Then by (4.17), we have

$$g_{r,s}(m, n) = as + \mu + n. \quad (4.24)$$

By (4.21), we have

$$f_{0,0}(m+r, 0) - f_{0,0}(m, 0) = \frac{r(\mu+1-b)}{\mu+1-a}.$$

Then by (4.13),

$$f_{r,0}(m, 0) = \frac{\mu+1-b}{\mu+1-a}(br + \lambda + m), \quad r \neq 0. \quad (4.25)$$

Setting  $h = r, r = s = k = n = 0$  in (3.7), we get

$$f_{r,0}(m, 0)\mu + f_{0,0}(m, 0)(\mu+1) - f_{0,0}(r+m, 0)\mu - f_{r,0}(m, 0)(\mu+1) = -r(a+\mu).$$

So

$$f_{r,0}(m, 0) = f_{0,0}(m, 0) - \frac{r\mu(\mu+1-b)}{\mu+1-a} + r(a+\mu).$$

Then by (4.25), we have

$$f_{0,0}(m, 0) = (\lambda+m) \frac{\mu+1-b}{\mu+1-a} + \frac{r(b-b^2-a+a^2)}{\mu+1-a}, \quad r \neq 0. \quad (4.26)$$

Therefore,  $b-b^2-a+a^2=0$ . Then

$$a=b \text{ or } a+b=1.$$

If  $a=b$ , then  $f_{0,0}(m, 0) = \lambda+m$ . By (4.20),

$$f_{0,0}(m, n) = \lambda+m. \quad (4.27)$$

In (3.7), let  $h = k = 0$ , then we have

$$\begin{aligned} & f_{0,0}(r+m, s+n)g_{r,s}(m, n) + g_{0,0}(m+r, s+n+1)f_{r,s}(m, n) \\ & - f_{r,s}(m, n)g_{0,0}(m, n) - g_{r,s}(m, n+1)f_{0,0}(m, n) \\ & = rg_{r,s+1}(m, n) + sf_{r,s}(m, n). \end{aligned} \quad (4.28)$$

By (4.24) and (4.28), we have

$$f_{r,s}(m, n) = ar + \lambda + m.$$

So we can deduce that

$$\begin{cases} f_{r,s}(m, n) = ar + \lambda + m, \\ g_{r,s}(m, n) = as + \mu + n. \end{cases}$$

If  $a+b=1$ , then

$$f_{0,0}(m, n) = \frac{\mu+n+a}{\mu+n+1-a}(\lambda+m). \quad (4.29)$$

By induction on  $n$ , we can deduce that

$$f_{0,1}(m, n) = \frac{(\mu + n + a)(\mu + n + 1 + a)}{(\mu + n + 1 - a)(\mu + n + 2 - a)}(\lambda + m)$$

and

$$f_{r,1}(m, n) = \frac{(\mu + n + a)(\mu + n + 1 + a)}{(\mu + n + 1 - a)(\mu + n + 2 - a)}((1 - a)r + \lambda + m). \quad (4.30)$$

On the other hand, by (4.27) and (4.28), we have

$$f_{r,1}(m, n) = ra + \frac{(\mu + n + a)}{(\mu + n + 2 - a)}[r(1 - 2a) + \frac{(\mu + n + 1 + a)}{(\mu + n + 1 - a)}(\lambda + m)]. \quad (4.31)$$

Then by (4.30) and (4.31), we have

$$a = 0, 1 \text{ or } \frac{1}{2}.$$

Then  $b = 1, 0$  or  $\frac{1}{2}$ . By the assumption,  $a \neq 0, 1$ , so  $a = \frac{1}{2} = b$ . This also means that

$$\begin{cases} f_{r,s}(m, n) = ar + \lambda + m, \\ g_{r,s}(m, n) = as + \mu + n. \end{cases}$$

(2) follows from the proof of (1).  $\square$

**Lemma 4.3.** *Suppose that  $\mu + n - a \neq 0$  for all  $n \in \mathbb{Z}$  and  $a = b = 1$ . Then we have*

$$\begin{cases} f_{r,s}(m, n) = r + \lambda + m, \\ g_{r,s}(m, n) = s + \mu + n, \end{cases} \quad (4.32)$$

or

$$\begin{cases} f_{r,s}(m, n) = \frac{\mu + n + s + 1}{\mu + n}(r + \lambda + m), \\ g_{r,s}(m, n) = \frac{\lambda + m + r}{\lambda + m}(s + \mu + n), \end{cases} \quad \lambda \notin \mathbb{Z}. \quad (4.33)$$

**Proof: Case 1.**  $\lambda \notin \mathbb{Z}$ . By the proof of (1) of Lemma 4.2 ( see (4.23) ), we know that

$$k_{r,m} = 1 \text{ or } k_{r,m} = \frac{r + \lambda + m}{\lambda + m}.$$

If  $k_{r,m} = 1$ , then by (4.17),  $g_{r,s}(m, n) = s + \mu + n$ . By (4.26) and (4.20), we have

$$f_{0,0}(m, n) = \lambda + m.$$

Then by (4.28), we have

$$f_{r,s}(m, n) = r + \lambda + m.$$

So we have (4.32). If  $k_{r,m} = \frac{r + \lambda + m}{\lambda + m}$ , then

$$g_{r,s}(m, n) = \frac{r + \lambda + m}{\lambda + m}(s + \mu + n). \quad (4.34)$$

By (4.22), we have

$$f_{0,0}(m + r, 0) - f_{0,0}(m, 0) = \frac{\mu + 1}{\mu}r.$$

By (4.13), we obtain

$$f_{r,0}(m, 0) = \frac{\mu + 1}{\mu}(r + \lambda + m), \quad r \neq 0.$$

In (3.7), let  $n = 0, k = r = s = 0$ , then we can deduce that

$$f_{0,0}(m, 0) = \frac{\mu + 1}{\mu}(\lambda + m).$$

Then by (4.20), we have

$$f_{0,0}(m, n) = \frac{\mu + n + 1}{\mu + n}(\lambda + m). \quad (4.35)$$

By (4.28), (4.34) and (4.35), we have (4.33).

**Case 2.**  $\lambda \in \mathbb{Z}$ . By (4.13), we have

$$f_{r,s}(-\lambda - r, n) = 0, \quad r \neq 0. \quad (4.36)$$

In (3.6), let  $h = -r, k = -1, m = -\lambda$ , then by (4.36), we have

$$f_{0,s}(-\lambda, n) = 0. \quad (4.37)$$

By (4.23), we have

$$k_{r,-\lambda-r} = 1, \quad r \neq 0.$$

Then by  $k_{r,-\lambda} = 1$  for all  $r \in \mathbb{Z}$  and (4.16), we have

$$k_{h+r,-\lambda-r} = k_{h,-\lambda}k_{r,-\lambda-r} = 1, \quad \forall h, r \in \mathbb{Z}.$$

So

$$k_{r,m} = 1, \quad \forall r, m \in \mathbb{Z}.$$

Then similar to the proof above, we have (4.32).  $\square$

**Lemma 4.4.** Suppose that  $\mu + n - a \neq 0$  for all  $n \in \mathbb{Z}$  and  $b = 1, a = 0$ . Then

$$\begin{cases} f_{r,s}(m, n) = r + \lambda + m, \\ g_{r,s}(m, n) = \frac{\lambda + m + r}{\lambda + m}(\mu + n), \end{cases} \quad \lambda \notin \mathbb{Z}, \quad (4.38)$$

or

$$\begin{cases} f_{r,s}(m, n) = \frac{\mu + n}{\mu + n + s + 1}(r + \lambda + m), \\ g_{r,s}(m, n) = \mu + n. \end{cases} \quad (4.39)$$

**Proof:** **Case 1.**  $\lambda \in \mathbb{Z}$ . As the proof in (2) of Lemma 4.3, we have

$$k_{r,m} = 1,$$

and

$$g_{r,s}(m, n) = \mu + n. \quad (4.40)$$

By (4.26) and (4.20), we have

$$f_{0,0}(m, n) = \frac{\mu + n}{\mu + n + 1}(\lambda + m).$$

Then by (4.28) and (4.40), we get (4.39).

**Case 2.**  $\lambda \notin \mathbb{Z}$ . Then

$$k_{r,m} = 1 \text{ or } k_{r,m} = \frac{\lambda + m + r}{\lambda + m}.$$

If  $k_{r,m} = 1$ , then we have (4.39). If  $k_{r,m} = \frac{\lambda + m + r}{\lambda + m}$ , then

$$g_{r,s}(m, n) = \frac{\lambda + m + r}{\lambda + m}(\mu + n). \quad (4.41)$$

Then we can deduce that

$$f_{0,0}(m, n) = \lambda + m.$$

By (4.41) and (4.28), we have (4.38).  $\square$

**Lemma 4.5.** *Suppose that  $\mu + n - a \notin \mathbb{Z}$  for all  $n \in \mathbb{Z}$  and  $b = 0, a = 1$ . Then*

$$\begin{cases} f_{r,s}(m, n) = \frac{\mu + n + s + 1}{\mu + n}(\lambda + m), \\ g_{r,s}(m, n) = s + \mu + n, \end{cases} \quad (4.42)$$

or

$$\begin{cases} f_{r,s}(m, n) = \lambda + m, \\ g_{r,s}(m, n) = \frac{\lambda + m}{\lambda + m + r}(s + \mu + n), \end{cases} \quad \lambda \notin \mathbb{Z}. \quad (4.43)$$

**Proof: Case 1.**  $\lambda \in \mathbb{Z}$ . As the proof above we have

$$k_{r,m} = 1$$

and

$$g_{r,s}(m, n) = s + \mu + n. \quad (4.44)$$

By (4.22), we have

$$f_{0,0}(m + r, 0) - f_{0,0}(m, 0) = \frac{\mu + 1}{\mu}r.$$

Then by (4.13), we obtain

$$f_{r,0}(m, 0) = \frac{\mu + 1}{\mu}(\lambda + m), \quad r \neq 0.$$

Let  $n = 0, k = r = s = 0$  in (3.7), then we have

$$f_{0,0}(m, 0) = \frac{\mu + 1}{\mu}(\lambda + m).$$

Then by (4.20), we get

$$f_{0,0}(m, n) = \frac{\mu + n + 1}{\mu + n}(\lambda + m).$$

Then by (4.44) and (4.28), we obtain (4.42).

**Case 2.** If  $\lambda \notin \mathbb{Z}$ , then

$$k_{r,m} = 1 \text{ or } k_{r,m} = \frac{\lambda + m}{\lambda + m + r}.$$

If  $k_{r,m} = 1$ , then  $g_{r,s}(m, n) = s + \mu + n$ . Using (4.26) and (4.20), we have

$$f_{0,0}(m, n) = \frac{\mu + n + 1}{\mu + n}(\lambda + m).$$

By (4.28), we have

$$f_{r,s}(m, n) = \frac{\mu + s + n + 1}{\mu + n}(\lambda + m).$$

Then we get (4.42). If  $k_{r,m} = \frac{\lambda + m}{\lambda + m + r}$ , then  $g_{r,s}(m, n) = \frac{\lambda + m}{\lambda + m + r}(s + \mu + n)$ . By (4.22), we have

$$f_{0,0}(m + r, 0) - f_{0,0}(m, 0) = r.$$

By (4.13), we obtain

$$f_{r,0}(m, 0) = \lambda + m, \quad r \neq 0.$$

Let  $k = r = s = n = 0, h \neq 0$  in (3.7), then we have

$$f_{0,0}(m, 0) = \lambda + m.$$

Then by (4.20), we have

$$f_{0,0}(m, n) = \lambda + m.$$

Using (4.28), we deduce

$$f_{r,s}(m, n) = \lambda + m.$$

So we have (4.43). □

Similarly we have the following lemma.

**Lemma 4.6.** *Suppose that  $\mu + n - a \neq 0$  for all  $n \in \mathbb{Z}$  and  $b = a = 0$ . Then*

$$\begin{cases} f_{r,s}(m, n) = \lambda + m, \\ g_{r,s}(m, n) = \mu + n, \end{cases}$$

or

$$\begin{cases} f_{r,s}(m, n) = \frac{\mu + n}{\mu + n + s + 1}(\lambda + m), \\ g_{r,s}(m, n) = \frac{\lambda + m}{\lambda + m + r}(\mu + n), \end{cases} \quad \lambda \notin \mathbb{Z}.$$

**Lemma 4.7.** *Suppose that  $\mu - a \in \mathbb{Z}$ . Then we also have  $b_n = b$  for all  $n \in \mathbb{Z}$ . Furthermore, we have*

$$\begin{cases} f_{r,s}(m, n) = ar + \lambda + m, \\ g_{r,s}(m, n) = as + \mu + n, \end{cases} \quad (4.45)$$

or

$$\begin{cases} f_{r,s}(m, n) = r + \lambda + m, \\ g_{r,s}(m, n) = \frac{\lambda + m + r}{\lambda + m}(\mu + n), \end{cases} \quad \lambda \notin \mathbb{Z}, \quad (4.46)$$

or

$$\begin{cases} f_{r,s}(m, n) = \lambda + m, \\ g_{r,s}(m, n) = \frac{\lambda + m}{\lambda + m + r}(s + \mu + n), \end{cases} \quad \lambda \notin \mathbb{Z}. \quad (4.47)$$

**Proof:** Let  $n_0 \in \mathbb{Z}$  be such that  $\mu + n_0 - a = 0$ . By (4.3), we have

$$(b_n r + \lambda + m)(\mu + n - a) = (b_{n-1} r + \lambda + m)(\mu + n - a).$$

So

$$b_n = \begin{cases} b_{n_0}, & n \geq n_0, \\ b_{n_0-1}, & n \leq n_0 - 1. \end{cases}$$

Then by (4.20), we have

$$f_{0,0}(m, n) = \lambda + m, \quad n \neq n_0 - 1.$$

Assume  $f_{0,0}(m, n_0 - 1) = \lambda + m + f(m)$  for all  $m \in \mathbb{Z}$ . Then

$$f_{0,0}(m, n) = \lambda + m + \delta_{n, n_0-1} f(m). \quad (4.48)$$

Since  $\mu + n - a \neq 0$  for all  $n \neq n_0$ , by (4.12) we can assume that

$$g_{r,-1}(m, n) = k_{r,m}(\mu + n - a), \quad \text{for } n \geq n_0,$$

$$g_{r,-1}(m, n) = k'_{r,m}(\mu + n - a), \quad \text{for } n \leq n_0 - 1.$$

Similar to the proof of Lemma 4.2, we have

$$k_{r,m} k_{-r,m+r} = 1, \quad k_{h,m+r} k_{r,m} = k_{r,m+h} k_{h,m} = k_{h+r,m}, \quad (4.49)$$

$$k'_{r,m}k'_{-r,m+r} = 1, \quad k'_{h,m+r}k'_{r,m} = k'_{r,m+h}k'_{h,m} = k'_{h+r,m}, \quad (4.50)$$

for all  $r, m, h \in \mathbb{Z}$ . By (4.11), we have

$$(s+2)g_{r,s}(m, n) = (as + \mu + n + a)g_{r,-1}(m, s + n + 1) - (as + \mu + n + a - 1)g_{r,-1}(m, n).$$

Then by (3.8), we have

$$\begin{aligned} g_{r,s}(m, n) &= k_{r,m}(as + \mu + n), \quad n \geq n_0, s \geq -1; \\ g_{r,s}(m, n) &= \frac{1}{s+2}[(as + \mu + n + a)(\mu + s + n + 1 - a)k'_{r,m} \\ &\quad - (as + \mu + n + a - 1)(\mu + n - a)k_{r,m}], \quad n \geq n_0, s \leq -3; \\ g_{r,s}(m, n) &= \frac{1}{s+2}[(as + \mu + n + a)(\mu + s + n + 1 - a)k_{r,m} \\ &\quad - (as + \mu + n + a - 1)(\mu + n - a)k'_{r,m}], \quad n \leq n_0 - 1, s \geq 0; \\ g_{r,s}(m, n) &= k'_{r,m}(as + \mu + n), \quad n \leq n_0 - 1, s \leq -3. \end{aligned}$$

Let  $h = 0, k = -3, s = 1$  in (3.8), then

$$g_{r,-2}(m, n) = \frac{1}{4}[(-3a + \mu + n + 1)g_{r,1}(m, n) - (-3a + \mu + n)g_{r,1}(m, n - 3)].$$

So we can deduce that

$$\begin{aligned} g_{r,-2}(m, n) &= k_{r,m}(-2a + \mu + n), \quad n \geq n_0 + 3; \\ g_{r,-2}(m, n) &= \frac{1}{12}[3(-3a + \mu + n)(2\mu + 2n + a - 1) + (2a + \mu + n)(\mu + n + 2 - a)]k_{r,m} \\ &\quad - \frac{1}{12}[3(-3a + \mu + n)(2\mu + 2n + a - 4) + (2a + \mu + n - 1)(\mu + n - a)]k'_{r,m}, \quad n \leq n_0 - 1; \\ g_{r,-2}(m, n_0) &= \frac{1}{2}ak_{r,m}; \\ g_{r,-2}(m, n_0 + 1) &= \frac{1}{2}[(1 - a)(2a + 1)k_{r,m} + (a - 1)(2a - 1)k'_{r,m}]; \\ g_{r,-2}(m, n_0 + 2) &= \frac{1}{6}[(a + 2)(-3a + 5)k_{r,m} + (a - 1)(3a - 2)k'_{r,m}]. \end{aligned}$$

If  $a = 1$ , letting  $k = 0, s = -2, n = n_0$  in (3.8), we have

$$k'_{h,m+r}k_{r,m} + k_{r,m+h}k_{h,m} = 2k_{r+h,m}.$$

Let  $r = 0$  and note  $k'_{0,m} = 1 = k_{0,m}$  for all  $m \in \mathbb{Z}$ , then we have

$$k'_{h,m} = k_{h,m}.$$

If  $a = 0$ , let  $k = -3, h = 0, s = -2, n = n_0 + 2$  in (3.8), then we obtain

$$g_{0,-3}(m + r, n_0)g_{r,-2}(m, n_0 + 2) - g_{r,-2}(m, n_0 - 1)g_{0,-3}(m + r, n_0 + 2) = g_{r,-5}(m, n_0 + 2).$$

Then we can also get  $k'_{r,m} = k_{r,m}$  for all  $r, m \in \mathbb{Z}$ . Therefore, if  $a = 0$  or  $1$ , we have

$$k'_{r,m} = k_{r,m}, \quad \forall r, m \in \mathbb{Z}.$$

Letting  $n = n_0 + 1, s = 0$  in (4.10), we have

$$(\lambda + m + ar)k_{r,m} = b_{n_0}r + \lambda + m. \quad (4.51)$$

Letting  $n = n_0 - 3, s = 0$  in (4.10), we obtain

$$(\lambda + m + ar)k'_{r,m} = b_{n_0-1}r + \lambda + m. \quad (4.52)$$

Replace  $r$  by  $-r$ ,  $m$  by  $m+r$  in (4.51) respectively, then we have

$$(\lambda + m + r - ar)k_{-r, m+r} = -b_{n_0}r + \lambda + m + r.$$

Similar to the proof of Lemma 4.2,

$$(k_{r,m} - 1)[(-b_{n_0}r + \lambda + m + r)k_{r,m} - (b_{n_0}r + \lambda + m)] = 0. \quad (4.53)$$

Furthermore, we can deduce that if  $b_{n_0} \neq 0, 1$ , then  $k_{r,m} = 1$  and  $b_{n_0} = a$ ; if  $b_{n_0} = 0$  or  $1$ , then  $k_{r,m} = 1$  or  $k_{r,m} = \frac{b_{n_0}r + \lambda + m}{-b_{n_0}r + \lambda + m + r}$ .

(1)  $\lambda \in \mathbb{Z}$ . If  $b_{n_0} = 0$ , let  $m = -\lambda$  in (4.51), then we have  $ark_{r,-\lambda} = 0$  for all  $r \in \mathbb{Z}$ . By the fact that  $k_{r,-\lambda} \neq 0$ , we have  $a = 0$ . If  $b_{n_0} = 1$ , we have  $(a-1)rk_{r,-\lambda-r} = 0$  for all  $r \in \mathbb{Z}$  by (4.51). Similarly we have  $a = 1$ . Therefore, if  $b_{n_0} = 0$  or  $1$  and  $\lambda \in \mathbb{Z}$ , we also have  $b_{n_0} = a$  and  $k_{r,m} = 1$ .

(2)  $\lambda \notin \mathbb{Z}$ . Then  $b_{n_0}r + \lambda + m \neq 0$ ,  $-b_{n_0}r + \lambda + m + r \neq 0$  for all  $r, m \in \mathbb{Z}$ . By the fact that  $k_{r,m} \neq 0$  for all  $r, m \in \mathbb{Z}$  and (4.51), we have  $ar + \lambda + m \neq 0$  and

$$k_{r,m} = \frac{b_{n_0}r + \lambda + m}{ar + \lambda + m}.$$

Obviously,  $b_{n_0} = a$  if  $k_{r,m} = 1$ . On the other hand, if  $k_{r,m} = \frac{b_{n_0}r + \lambda + m}{-b_{n_0}r + \lambda + m + r}$ , then we have

$$\frac{b_{n_0}r + \lambda + m}{-b_{n_0}r + \lambda + m + r} = \frac{b_{n_0}r + \lambda + m}{ar + \lambda + m}.$$

It is easy to see that  $a = 1 - b_{n_0}$  and

$$\begin{cases} k_{r,m} = \frac{\lambda + m}{\lambda + m + r}, & b_{n_0} = 0; \\ k_{r,m} = \frac{\lambda + m + r}{\lambda + m}, & b_{n_0} = 1. \end{cases}$$

For  $k'_{r,m}$  and  $b_{n_0-1}$  we have the similar results. So we have

(i) if  $a \neq 0, 1$  or  $\lambda \in \mathbb{Z}$ , then  $b_{n_0} = a = b_{n_0-1}$  and  $k_{r,m} = k'_{r,m} = 1$ ;

(ii) if  $a = 0$  or  $1$  and  $\lambda \notin \mathbb{Z}$ , then  $k_{r,m} = k'_{r,m}$  and

(a) if  $k_{r,m} = 1$ , then  $b_{n_0} = b_{n_0-1} = a$ ,

(b) if  $k_{r,m} = \frac{\lambda + m}{\lambda + m + r}$ , then  $b_{n_0} = b_{n_0-1} = 0, a = 1$ ,

(c) if  $k_{r,m} = \frac{\lambda + m + r}{\lambda + m}$ , then  $b_{n_0} = b_{n_0-1} = 1, a = 0$ .

Therefore, we always have  $k_{r,m} = k'_{r,m}$  and  $b_n = b$  for all  $r, m, n \in \mathbb{Z}$ . Furthermore,

$$g_{r,s}(m, n) = k_{r,m}(as + \mu + n),$$

for all  $r, s, m, n \in \mathbb{Z}$ .

**Case 1.** If  $k_{r,m} = 1$ , then  $b = a$  and  $g_{r,s}(m, n) = as + \mu + n$ . Then by (4.28), we have

$$\begin{aligned} f_{r,0}(m, n_0 - 1) &= ar + \lambda + m + af(m) + (1-a)f(m+r); \\ f_{r,s}(m, n_0 - 1) &= ar + \lambda + m + a(s+1)f(m), \quad s \neq 0; \\ f_{r,s}(m, n_0 - s - 1) &= ar + \lambda + m + (1-a)(s+1)f(m+r), \quad s \neq 0; \\ f_{r,s}(m, n) &= ar + \lambda + m, \quad n \neq n_0 - 1, n \neq n_0 - s - 1. \end{aligned}$$

Let  $h = 0, k = -2, s = 0, n = n_0 - 1$  in (3.6), then we have

$$f_{0,-2}(m+r, n_0)f_{r,0}(m, n_0-1) - f_{r,0}(m, n_0-2)f_{0,-2}(m, n_0-1) = rf_{r,-1}(m, n_0-1). \quad (4.54)$$



Then

$$a[(a+1)r+2\lambda+2m]f(m)+(1-a)(\lambda+m+r)f(m+r)=0. \quad (4.55)$$

Let  $r=0$  in (4.55), then we have

$$(a+1)(\lambda+m)f(m)=0. \quad (4.56)$$

Let  $h=0, k=1, s=0, n=n_0-1$  in (3.6), and we have

$$f_{0,1}(m+r, n_0)f_{r,0}(m, n_0-1)-f_{r,0}(m, n_0+1)f_{0,1}(m, n_0-1)=rf_{r,2}(m, n_0-1). \quad (4.57)$$

Then

$$[(3a-2)(\lambda+m)+(2a-4)ar]f(m)-(a-1)(\lambda+m+r)f(m+r)=0. \quad (4.58)$$

Let  $r=0$  in (4.58), then

$$(2a-1)(\lambda+m)f(m)=0. \quad (4.59)$$

By (4.56) and (4.59), we obtain

$$(\lambda+m)f(m)=0. \quad (4.60)$$

Then by (4.55) and (4.58), we know

$$af(m)=0. \quad (4.61)$$

It is easy to see that if  $a \neq 0$  or  $\lambda \notin \mathbb{Z}$ , we have  $f(m)=0$  for all  $m \in \mathbb{Z}$ . If  $\lambda \in \mathbb{Z}$  and  $a=0$ , by (4.60), we have  $f(m)=0$  for all  $m \neq -\lambda$ . Let  $h=k=-1, r=1, s=0, m=-\lambda, n=n_0-1$  in (3.6), then we have  $f(-\lambda)=0$ . Therefore,

$$f(m)=0,$$

for all  $m \in \mathbb{Z}$ . We have (4.45).

**Case 2.** If  $k_{r,m} = \frac{\lambda+m+r}{\lambda+m}$ , then  $b=1, a=0$  and  $g_{r,s}(m, n) = \frac{\lambda+m+r}{\lambda+m}(\mu+n)$ , where  $\lambda \notin \mathbb{Z}$ . Note that  $\mu+n_0=0$ . By (4.28), we have

$$f_{r,s}(m, n) = r + \lambda + m, \quad n \neq n_0 - 1, s + n \neq n_0 - 1;$$

$$f_{r,s}(m, n_0 - 1) = r + \lambda + m, \quad s \neq 0;$$

$$f_{r,0}(m, n_0 - 1) = r + \lambda + m + \frac{\lambda + m + r}{\lambda + m}f(m + r);$$

$$f_{r,s}(m, n_0 - s - 1) = r + \lambda + m + (s + 1)\frac{\lambda + m + r}{\lambda + m}f(m + r), \quad s \neq 0.$$

By (4.54), we have

$$(\lambda + m + r)\frac{\lambda + m + r}{\lambda + m}f(m + r) = r(\lambda + m + r).$$

Let  $r=0$ , since  $\lambda \notin \mathbb{Z}$ , we also have  $f(m)=0$  for all  $m \in \mathbb{Z}$ . Therefore, (4.46) holds.

**Case 3.** If  $k_{r,m} = \frac{\lambda+m}{\lambda+m+r}$ , then  $b=0, a=1$  and  $g_{r,s}(m, n) = \frac{\lambda+m}{\lambda+m+r}(s+\mu+n)$ , where  $\lambda \notin \mathbb{Z}$ . Note that  $\mu+n_0=1$ . By (4.28), we have

$$f_{r,s}(m, n) = \lambda + m, \quad n \neq n_0 - 1, s + n \neq n_0 - 1;$$

$$f_{r,s}(m, n_0 - s - 1) = \lambda + m, \quad s \neq 0;$$

$$f_{r,s}(m, n_0 - 1) = \lambda + m + (s + 1)\frac{\lambda + m}{\lambda + m + r}f(m), \quad s \neq 0;$$

$$f_{r,0}(m, n_0 - 1) = \lambda + m + \frac{\lambda + m}{\lambda + m + r}f(m).$$

By (4.54), we have

$$(\lambda + m + r)[\lambda + m + \frac{\lambda + m}{\lambda + m + r}f(m)] - (\lambda + m)[\lambda + m - \frac{\lambda + m}{\lambda + m + r}f(m)] = r(\lambda + m).$$

Hence,  $(\lambda + m)f(m) = 0$ . Since  $\lambda \notin \mathbb{Z}$ , we have  $f(m) = 0$  for all  $m \in \mathbb{Z}$ . Therefore, we have (4.47).  $\square$

### 5. Proof of Theorem 3.4 for the other cases

In this section we prove that there are no other situations except the seven ones in Theorem 3.4.

**Lemma 5.1.** *Suppose that*

$$\begin{aligned} Q_{0,s}v_{m,n} &= (a_ms + \mu + n)v_{m,n+s}, \\ P_{r,-1}v_{m,n} &= (b_nr + m)v_{m+r,n}, \quad m \neq 0, m+r \neq 0, \\ P_{r,-1}v_{m,n_0} &= (r+m)v_{m+r,n_0}, \quad m \neq 0, \\ P_{r,-1}v_{0,n_0} &= r(1 + (r+1)b'_{n_0})v_{r,n_0}, \end{aligned}$$

for some  $n_0 \in \mathbb{Z}$ . Then  $b'_{n_0} = 0$ .

**Proof:** By (4.3), we have

$$(\mu + n - a_{m+r})f_{r,-1}(m, n) = (\mu + n - a_m)f_{r,-1}(m, n-1). \quad (5.1)$$

Let  $m \neq 0, m+r \neq 0$  in (5.1), we have

$$(\mu + n - a_{m+r})(b_nr + m) = (\mu + n - a_m)(b_{n-1}r + m), \quad m \neq 0, m+r \neq 0. \quad (5.2)$$

Replace  $r$  by  $-r$ ,  $m$  by  $m+r$  in (5.2) respectively, then we have

$$(\mu + n - a_m)(-b_nr + m + r) = (\mu + n - a_{m+r})(-b_{n-1}r + m + r), \quad m \neq 0, m+r \neq 0. \quad (5.3)$$

From (5.2) and (5.3), we have

$$(b_n + b_{n-1} - 1)(a_{m+r} - a_m) = 0, \quad \text{for all } m \neq 0, m+r \neq 0. \quad (5.4)$$

Note that  $b_{n_0} = 1$ . Similar to the proof of Lemma 4.1 we have  $a_{m+r} = a_m$  for all  $m \neq 0, m+r \neq 0$ . Therefore,

$$a_m = a, \quad m \neq 0.$$

By (5.2), we have

$$(\mu + n - a)(b_n - b_{n-1}) = 0. \quad (5.5)$$

Let  $m = 0, r \neq 0$  in (5.1), then we have

$$(\mu + n - a)f_{r,-1}(0, n) = (\mu + n - a_0)f_{r,-1}(0, n-1). \quad (5.6)$$

**Case 1.** If  $\mu + n - a \neq 0$  for all  $n \in \mathbb{Z}$ , then by (5.5), we have  $b_n = b_{n_0} = 1$  for all  $n \in \mathbb{Z}$ . Then

$$f_{r,-1}(m, n) = r + m, \quad m \neq 0, m+r \neq 0.$$

On the other hand, for each  $n \in \mathbb{Z}$ ,  $V_n^R = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}v_{m,n}$  is a module of intermediate series over  $\mathcal{P}_{-1}$ , so we have

$$\begin{cases} f_{r,-1}(m, n) = r + m, & m \neq 0, \\ f_{r,-1}(0, n) = r(1 + (r+1)b'_n). \end{cases}$$

By (5.6), we have

$$(\mu + n - a)r(1 + (r+1)b'_n) = (\mu + n - a_0)r(1 + (r+1)b'_{n-1}),$$

for all  $r, n \in \mathbb{Z}$ . Therefore  $a_0 = a, b'_n = b'_{n-1}$ , and

$$a_m = a, \quad b'_n = b',$$

for all  $m, n \in \mathbb{Z}$ . Similar to the proof of Lemma 4.2, we have

$$\begin{aligned} g_{r,s}(m, n) &= k_{r,m}(as + \mu + n), \\ (\mu + n + 1 - a)f_{0,0}(m, n) &= nm + (\mu + 1 - a)f_{0,0}(m, 0), \end{aligned} \quad (5.7)$$

$$k_{r,m}[(\mu + 2 - a)f_{0,0}(m, 1) - (\mu + 1 - a)f_{0,0}(m + r, 0) + r(\mu + 1)] = f_{r,-1}(m, 1).$$

By (5.7), we have

$$k_{r,m}(\mu + 1 - a)(f_{0,0}(m, 0) - f_{0,0}(m + r, 0)) + k_{r,m}(m + r + r\mu) = f_{r,-1}(m, 1). \quad (5.8)$$

Replace  $r$  by  $-r$ ,  $m$  by  $m + r$  in (5.8) respectively, then we obtain

$$k_{-r,m+r}(\mu + 1 - a)(f_{0,0}(m + r, 0) - f_{0,0}(m, 0)) + k_{-r,m+r}(m - r\mu) = f_{-r,-1}(m + r, 1).$$

According to  $k_{r,m}k_{-r,m+r} = 1$ , we have

$$k_{r,m}(\mu + 1 - a)(f_{0,0}(m + r, 0) - f_{0,0}(m, 0)) + k_{r,m}(m - r\mu) = f_{-r,-1}(m + r, 1)k_{r,m}^2.$$

By (5.8), we get

$$f_{-r,-1}(m + r, 1)k_{r,m}^2 - (2m + r)k_{r,m} + f_{r,-1}(m, 1) = 0. \quad (5.9)$$

Let  $m \neq 0, m + r \neq 0$  in (5.9), then

$$mk_{r,m}^2 - (2m + r)k_{r,m} + r + m = 0, \quad m \neq 0, m + r \neq 0.$$

So

$$k_{r,m} = 1 \text{ or } k_{r,m} = \frac{r + m}{m}, \quad m \neq 0, m + r \neq 0. \quad (5.10)$$

Let  $m = 0, r \neq 0$  in (5.9), then we obtain

$$k_{r,0} = 1 + (r + 1)b', \quad r \neq 0. \quad (5.11)$$

By  $k_{r,0}k_{-r,r} = 1$ , we have  $k_{-r,r} = \frac{1}{1 + (r + 1)b'}$  for  $r \neq 0$ . By the fact that  $k_{h+r,m} = k_{h,m+r}k_{r,m}$  for all  $h, r, m \in \mathbb{Z}$ , let  $m = -r, h \neq 0, r \neq 0$ , then if  $k_{r,m} = 1$  for  $m \neq 0, m + r \neq 0$  we have

$$1 = \frac{1}{1 + (1 - r)b'}[1 + (h + 1)b'].$$

Then

$$(h + r)b' = 0 \text{ for all } h \neq 0, r \neq 0.$$

Therefore,  $b' = 0$ . If  $k_{r,m} = \frac{r + m}{m}$  for  $m \neq 0, m + r \neq 0$ , then we have

$$-\frac{h}{r} = \frac{1}{1 + (1 - r)b'}(1 + (h + 1)b'), \quad h \neq 0, r \neq 0.$$

So  $b' = -1$ , and

$$\begin{cases} k_{r,m} = \frac{r + m}{m}, & m \neq 0, m + r \neq 0, \\ k_{r,0} = -r, & r \neq 0, \\ k_{r,-r} = \frac{1}{r}, & r \neq 0, \end{cases}$$

and

$$\begin{cases} f_{r,-1}(m, n) = r + m, & m \neq 0, \\ f_{r,-1}(0, n) = -r^2. \end{cases}$$

Using (5.8), we have

$$\begin{aligned} f_{0,0}(r, 0) - f_{0,0}(0, 0) &= \frac{\mu}{\mu + 1 - a} r, \\ f_{0,0}(m + r, 0) - f_{0,0}(m, 0) &= \frac{\mu + 1}{\mu + 1 - a} r, \quad m \neq 0, m + r \neq 0. \end{aligned}$$

Then by (4.9), we obtain

$$\begin{aligned} f_{r,0}(0, 0) &= \frac{\mu}{\mu + 1 - a} r, \quad r \neq 0, \\ f_{r,0}(m, 0) &= \frac{\mu + 1}{\mu + 1 - a} (r + m), \quad m \neq 0, r \neq 0. \end{aligned}$$

Let  $h = r, r = s = k = n = 0$  in (3.7), then we have

$$f_{r,0}(m, 0) = k_{r,m}[(\mu + 1)f_{0,0}(m, 0) - \mu f_{0,0}(m + r, 0) + r(a + \mu)].$$

Therefore

$$f_{0,0}(m, 0) = k_{-r,m+r} f_{r,0}(m, 0) + \mu[f_{0,0}(m + r, 0) - f_{0,0}(m, 0)] - r(a + \mu), \quad (5.12)$$

and

$$\begin{aligned} f_{0,0}(0, 0) &= -\frac{\mu}{\mu + 1 - a} - \frac{\mu - a(a - 1)}{\mu + 1 - a} r, \\ f_{0,0}(m, 0) &= \frac{\mu + 1}{\mu + 1 - a} m + \frac{a(a - 1)}{\mu + 1 - a} r, \quad m \neq 0. \end{aligned}$$

This forces  $a = 0$  or  $a = 1$  and  $\mu = 0$ . Thus  $\mu + a = 0$  or  $\mu - 1 + a = 0$ , a contradiction.

**Case 2.** If there exists  $n_1 \in \mathbb{Z}$  such that  $\mu + n_1 - a = 0$ , then by (5.5) we have

$$b_n = \begin{cases} b_{n_1}, & n \geq n_1, \\ b_{n_1-1}, & n \leq n_1 - 1. \end{cases}$$

Without loss of generality, we may assume  $n_0 \geq n_1$ , then  $b_n = 1$  for  $n \geq n_1$ . By the results on representations of the Virasoro algebra, we have

$$\begin{cases} f_{r,-1}(m, n) = r + m, & m \neq 0, \\ f_{r,-1}(0, n) = r(1 + (r + 1)b'_n), \end{cases} \quad n \geq n_1.$$

In (5.6), letting  $m = 0, r \neq 0, n = n_0 + 1 = n_1 + n'$ , where  $n' > 0$ , we have

$$n' f_{r,-1}(0, n_0 + 1) = (a - a_0 + n') f_{r,-1}(0, n_0).$$

Then  $n' r(1 + (r + 1)b'_{n_0+1}) = (a - a_0 + n') r(1 + (r + 1)b'_{n_0})$  for all  $r \in \mathbb{Z}$ . So,

$$b'_{n_0+1} = b'_{n_0}, \quad a_0 = a.$$

Therefore  $a_m = a$  for all  $m \in \mathbb{Z}$ . By (5.6), we have

$$f_{r,-1}(m, n) = \begin{cases} f_{r,-1}(m, n_1), & n \geq n_1, \\ f_{r,-1}(m, n_1 - 1), & n \leq n_1 - 1, \end{cases}$$

and

$$\begin{cases} f_{r,-1}(m, n) = r + m, & m \neq 0, \\ f_{r,-1}(0, n) = r(1 + (r + 1)b'_{n_0}), \end{cases} \quad n \geq n_1.$$

Similar to the proof of Lemma 4.7, we assume

$$g_{r,-1}(m, n) = k_{r,m}(\mu + n - a), \quad \text{for } n \geq n_1.$$

Letting  $h = k = 0, s = -1, m = -r \neq 0, n \geq n_1$  in (3.6), we have

$$f_{0,0}(0, n)f_{r,-1}(-r, n) - f_{r,-1}(-r, n+1)f_{0,0}(-r, n) = rf_{0,0}(-r, n).$$

Then we can deduce that

$$f_{0,0}(-r, n) = 0, \quad r \neq 0, n \geq n_1.$$

Letting  $h = -r \neq 0, k = 0, s = -1, m = 0, n \geq n_1$  in (3.6), we obtain

$$f_{-r,0}(r, n)f_{r,-1}(0, n) - f_{r,-1}(-r, n+1)f_{-r,0}(0, n) = 2rf_{0,0}(0, n).$$

Then

$$f_{0,0}(0, n) = 0, \quad n \geq n_1.$$

Therefore,

$$f_{r,0}(-r, n) = 0, \quad n \geq n_1.$$

Letting  $h = k = 0, s = 0, m = 0, n = n_1 + 1$  in (3.7), we obtain

$$f_{0,0}(r, n_1 + 1)g_{r,0}(0, n_1 + 1) + f_{r,0}(0, n_1 + 1) = rg_{r,1}(0, n_1 + 1). \quad (5.13)$$

By (4.11), we get

$$\begin{aligned} g_{r,0}(0, n_1 + 1) &= (a + 1)k_{r,0}, \quad g_{r,1}(0, n_1 + 1) = (2a + 1)k_{r,0}, \\ g_{-r,-1}(r, n_1 + 2) &= 2k_{-r,r}, \quad g_{-r,0}(r, n_1 + 1) = (a + 1)k_{-r,r}. \end{aligned}$$

By (5.13), we have

$$f_{0,0}(r, n_1 + 1)(a + 1)k_{r,0} + f_{r,0}(0, n_1 + 1) = r(2a + 1)k_{r,0}. \quad (5.14)$$

Letting  $h = k = 0, s = -1, m = 0, n = n_1 + 1$  in (3.6), we have

$$f_{0,0}(r, n_1 + 1)f_{r,-1}(0, n_1 + 1) - f_{r,-1}(0, n_1 + 2)f_{0,0}(0, n_1 + 1) = rf_{r,0}(0, n_1 + 1).$$

So

$$r(1 + (r + 1)b'_{n_0})f_{0,0}(r, n_1 + 1) = rf_{r,0}(0, n_1 + 1),$$

and therefore

$$f_{r,0}(0, n_1 + 1) = (1 + (r + 1)b'_{n_0})f_{0,0}(r, n_1 + 1), \quad r \neq 0. \quad (5.15)$$

Letting  $h = -r, k = -1, r = 0, s = 0, m = r \neq 0, n = n_1 + 1$  in (3.7), we have

$$g_{-r,-1}(r, n_1 + 2)f_{0,0}(r, n_1 + 1) = rg_{-r,0}(r, n_1 + 1).$$

Then we obtain

$$f_{0,0}(r, n_1 + 1) = \frac{a + 1}{2}r, \quad r \neq 0. \quad (5.16)$$

So we have  $f_{0,0}(m, n_1 + 1) = \frac{a + 1}{2}m$  for all  $m \in \mathbb{Z}$ , and

$$f_{r,0}(0, n_1 + 1) = \frac{a + 1}{2}(1 + (r + 1)b'_{n_0})r, \quad r \neq 0.$$

On the other hand,

$$f_{0,0}(r, n_1 + 1)(a + 1)k_{r,0} + f_{r,0}(0, n_1 + 1) = (2a + 1)rk_{r,0}.$$

Then we have

$$(2a + 1 - a^2)k_{r,0} = (a + 1)(1 + (r + 1)b'_{n_0}), \quad r \neq 0.$$

Letting  $h = 0, k = -1, r = 0, s = 0, n = n_1 + 1$  in (3.7), we have

$$2f_{0,0}(m, n_1 + 1) - f_{0,0}(m, n_1) = m. \quad (5.17)$$

Then

$$f_{0,0}(m, n_1) = am.$$

Let  $s = 0, n = n_1 + 1$  in (4.10), then we get

$$k_{r,m}[2f_{0,0}(m, n_1 + 1) - f_{0,0}(m + r, n_1) + r(a + 1)] = f_{r,-1}(m, n_1 + 1).$$

So

$$(r + m)k_{r,m} = f_{r,-1}(m, n_1 + 1). \quad (5.18)$$

Let  $m \neq 0$  and  $m = 0$  in (5.18) respectively, then we have

$$\begin{aligned} k_{r,m} &= 1, \quad m \neq 0, \quad m + r \neq 0; \\ k_{r,0} &= 1 + (r + 1)b'_{n_0}, \quad r \neq 0. \end{aligned}$$

Similar to the proof above, we have  $b'_{n_0} = 0$ .  $\square$

The proof of the following lemma is similar that of Lemma 5.1.

**Lemma 5.2.** *Suppose that*

$$\begin{aligned} Q_{0,s}v_{m,n} &= (a_ms + \mu + n)v_{m,n+s}, \\ P_{r,-1}v_{m,n} &= (b_nr + m)v_{m+r,n}, \quad m \neq 0, m \neq -r, \\ P_{r,-1}v_{m,n_0} &= mv_{m+r,n_0}, \quad m \neq -r, \\ P_{r,-1}v_{-r,n_0} &= -r(1 + (r + 1)a')v_{0,n_0}, \end{aligned}$$

for some  $n_0 \in \mathbb{Z}$ . Then  $a' = 0$ .

**Lemma 5.3.** *Suppose that*

$$\begin{aligned} Q_{0,s}v_{m,n} &= (s + n)v_{m,n+s}, \quad n \neq 0, \\ Q_{0,s}v_{m,0} &= s(1 + (s + 1)a'_m)v_{m,s}, \\ P_{r,-1}v_{m,n} &= (b_nr + \lambda + m)v_{m+r,n}. \end{aligned}$$

Then  $a'_m = 0$  for all  $m \in \mathbb{Z}$ .

**Proof:** By (4.3), we have  $(n - 1)f_{r,-1}(m, n) = (n - 1)f_{r,-1}(m, n - 1)$ . Then

$$f_{r,-1}(m, n) = f_{r,-1}(m, n - 1), \quad n \neq 1.$$

That is,

$$b_n = \begin{cases} b_1, & n \geq 1, \\ b_0, & n \leq 0. \end{cases}$$

Letting  $h = r, k = -1, r = 0, s = -1$  in (3.8), we have

$$(n - 1)g_{r,-1}(m, n - 1) = (n - 2)g_{r,-1}(m, n). \quad (5.19)$$

Similar to the proof of Lemma 4.7, we can assume

$$\begin{aligned} g_{r,-1}(m, n) &= l_{r,m}(n - 1), \quad n \geq 1, \\ g_{r,-1}(m, n) &= l'_{r,m}(n - 1), \quad n \leq 0. \end{aligned}$$

Then we deduce that  $l_{r,m}, l'_{r,m}$  have the same properties as  $k_{r,m}, k'_{r,m}$ :

$$l_{r,m}l_{-r,m+r} = 1, \quad l_{h,m+r}l_{r,m} = l_{r,m+h}l_{h,m} = l_{h+r,m}, \quad (5.20)$$

$$l'_{r,m}l'_{-r,m+r} = 1, \quad l'_{h,m+r}l'_{r,m} = l'_{r,m+h}l'_{h,m} = l'_{h+r,m}. \quad (5.21)$$

As the proof of Lemma 4.7, we have  $l'_{r,m} = l_{r,m}$ , for all  $r, m \in \mathbb{Z}$  and

$$\begin{cases} g_{r,s}(m, n) = l_{r,m}(s + n), & n \neq 0; \\ g_{r,s}(m, 0) = l_{r,m}s(1 + (s + 1)a'_m). \end{cases} \quad (5.22)$$

Let  $h = 0, k = -1, r = 0, s = 0$  in (3.7), then we have

$$f_{0,0}(m, n) = \lambda + m, \quad n \neq 0.$$

Assume  $f_{0,0}(m, 0) = \lambda + m + f(m)$  for all  $m \in \mathbb{Z}$ . Then

$$f_{0,0}(m, n) = \lambda + m + \delta_{n,0}f(m). \quad (5.23)$$

Let  $h = r, k = -1, r = 0, s = 0$  in (3.7), then we obtain

$$nl_{r,m}f_{0,0}(m, n) - (n - 1)l_{r,m}f_{0,0}(m + r, n - 1) = -rl_{r,m}n + b_nr + \lambda + m. \quad n \neq 0.$$

By (5.23), we have

$$nl_{r,m}(\lambda + m + \delta_{n,0}f(m)) - (n - 1)l_{r,m}(\lambda + m + r + \delta_{n,1}f(m + r)) = -rl_{r,m}n + b_nr + \lambda + m.$$

Then

$$(\lambda + m + r)l_{r,m} = b_nr + \lambda + m. \quad (5.24)$$

Replacing  $r$  by  $-r$ ,  $m$  by  $m + r$  in (5.24) respectively, we have

$$(\lambda + m)l_{-r,m+r} = -b_nr + \lambda + m + r.$$

By (5.20), we get

$$(-b_nr + \lambda + m + r)l_{r,m} = \lambda + m. \quad (5.25)$$

Using (5.24) and (5.26), we obtain

$$b_nl_{r,m} = b_n, \quad r \neq 0. \quad (5.26)$$

Then  $l_{r,m} = 1$  for  $r \neq 0$  or  $b_n = 0$  for all  $n \in \mathbb{Z}$ . If  $l_{r,m} = 1$  for  $r \neq 0$ , then  $l_{r,m} = 1$  for all  $r, m \in \mathbb{Z}$ . By (5.24), we have  $b_n = 1$  for all  $n \in \mathbb{Z}$ . If  $b_n = 0$  for all  $n \in \mathbb{Z}$ , by (5.24) we have

$$(\lambda + m + r)l_{r,m} = \lambda + m.$$

It is easy to see that  $\lambda \notin \mathbb{Z}$  and  $l_{r,m} = \frac{\lambda + m}{\lambda + m + r}$ .

**Case 1.** If  $l_{r,m} = 1$  and  $b_n = 1$  for all  $r, m, n \in \mathbb{Z}$ , then

$$\begin{cases} g_{r,s}(m, n) = s + n, & n \neq 0, \\ g_{r,s}(m, 0) = s(1 + (s + 1)a'_m), \\ f_{r,-1}(m, n) = r + \lambda + m. \end{cases}$$

By (4.28), we have

$$f_{r,s}(m, n) = rg_{r,s+1}(m, n) + g_{r,s}(m, n + 1)f_{0,0}(m, n) - g_{r,s}(m, n)f_{0,0}(m + r, s + n). \quad (5.27)$$

Then we can deduce that

$$\begin{aligned} f_{r,s}(m, n) &= r + \lambda + m, \quad n \neq 0, -1, -s; \\ f_{r,s}(m, 0) &= r + \lambda + m + (s + 1)f(m) + (s + 1)(2r - s(\lambda + m))a'_m; \\ f_{r,s}(m, -1) &= r + \lambda + m + s(s + 1)(\lambda + m)a'_m; \\ f_{r,s}(m, -s) &= r + \lambda + m, \quad n \neq 0, -1. \end{aligned}$$

Letting  $h = 0, k = -2, s = 0$  in (3.6), we obtain

$$f_{0,-2}(m + r, n + 1)f_{r,0}(m, n) - f_{r,0}(m, n - 1)f_{0,-2}(m, n) = rf_{r,-1}(m, n). \quad (5.28)$$

Let  $n = 0$  and  $n = 1$  in (5.28) respectively, then we obtain

$$(\lambda + m + r)f(m) + (\lambda + m + r)^2 a'_m = 0, \quad (5.29)$$

$$(\lambda + m)(f(m) + 2ra'_m) = 0. \quad (5.30)$$

By (5.30), we have

$$(\lambda + m)f(m) = 0, \quad (\lambda + m)a'_m = 0.$$

If  $\lambda \notin \mathbb{Z}$ , then  $f(m) = 0$ ,  $a'_m = 0$  for all  $m \in \mathbb{Z}$ . If  $\lambda \in \mathbb{Z}$ , then  $f(m) = 0$ ,  $a'_m = 0$  for all  $m \neq -\lambda$ . Let  $m = -\lambda$  in (5.29), then

$$f(-\lambda) + ra'_{-\lambda} = 0, \quad (5.31)$$

for all  $r \neq 0$ . So

$$f(-\lambda) = 0, \quad a'_{-\lambda} = 0,$$

and therefore,  $a'_m = 0$  for all  $m \in \mathbb{Z}$ .

**Case 2.** If  $l_{r,m} = \frac{\lambda + m}{\lambda + m + r}$  and  $b_n = 0$  for all  $r, m, n \in \mathbb{Z}$ , where  $\lambda \notin \mathbb{Z}$ , then

$$\begin{cases} g_{r,s}(m, n) = \frac{\lambda + m}{\lambda + m + r}(s + n), & n \neq 0, \\ g_{r,s}(m, 0) = \frac{\lambda + m}{\lambda + m + r}s(1 + (s + 1)a'_m), \\ f_{r,-1}(m, n) = \lambda + m. \end{cases}$$

By (5.27), we have

$$\begin{aligned} f_{r,s}(m, n) &= \lambda + m, \quad n \neq 0, -1, -s; \\ f_{r,s}(m, 0) &= \lambda + m + \frac{s + 1}{\lambda + m + r}[f(m) + r(s + 2)a'_m]; \\ f_{r,s}(m, -1) &= \lambda + m + \frac{\lambda + m}{\lambda + m + r}s(s + 1)a'_m; \\ f_{r,s}(m, -s) &= \lambda + m, \quad n \neq 0, -1. \end{aligned}$$

Let  $n = 1$  in (5.28) and note that  $\lambda \notin \mathbb{Z}$ , then we have  $2ra'_m + f(m) = 0$  for all  $m, r \in \mathbb{Z}$ . Therefore, we have

$$f(m) = 0, \quad a'_m = 0, \quad \forall m \in \mathbb{Z}.$$

□

**Lemma 5.4.** *Suppose that*

$$\begin{aligned} Q_{0,s}v_{m,n} &= (s + n)v_{m,n+s}, \quad n \neq 0, \\ Q_{0,s}v_{m,0} &= s(1 + (s + 1)a'_m)v_{m,s}, \\ P_{r,-1}v_{m,n} &= (b_n r + m)v_{m+r,n}, \quad m \neq 0, m + r \neq 0, \\ P_{r,-1}v_{m,n_0} &= (r + m)v_{m+r,n_0}, \quad m \neq 0, \\ P_{r,-1}v_{0,n_0} &= r(1 + (r + 1)b'_{n_0})v_{r,n_0}, \end{aligned}$$

for some  $n_0 \in \mathbb{Z}$ . Then  $b'_{n_0} = 0$ ,  $a'_m = 0$ ,  $b_n = 1$  for all  $m, n \in \mathbb{Z}$ .

**Proof:** By Lemma 5.3, we have

$$\begin{cases} g_{r,s}(m, n) = l_{r,m}(s + n), & n \neq 0, \\ g_{r,s}(m, 0) = l_{r,m}s(1 + (s + 1)a'_m), \end{cases}$$



and

$$f_{r,-1}(m, n) = \begin{cases} f_{r,-1}(m, 1), & n \geq 1, \\ f_{r,-1}(m, 0), & n \leq 0. \end{cases}$$

Note that  $b_{n_0} = 1$ . Without loss of generality, we assume  $n_0 \geq 1$ , then  $b_1 = 1$  and

$$\begin{cases} f_{r,-1}(m, n) = r + m, & m \neq 0, \\ f_{r,-1}(m, n) = r(1 + (r + 1)b'_{n_0}), & n \geq 1. \end{cases}$$

Similar to the proof of Lemma 5.3, we have

$$(m + r)l_{r,m} = f_{r,-1}(m, n). \quad (5.32)$$

Let  $n \geq 1$  in (5.32), then we obtain

$$\begin{aligned} (m + r)l_{r,m} &= m + r, \quad m \neq 0, \\ rl_{r,0} &= r(1 + (r + 1)b'_{n_0}). \end{aligned}$$

Hence,

$$\begin{aligned} l_{r,m} &= 1, \quad m \neq 0, \quad m + r \neq 0; \\ l_{r,0} &= (1 + (r + 1)b'_{n_0}), \quad r \neq 0. \end{aligned}$$

Similar to the proof above, we have  $b'_{n_0} = 0$  and  $l_{r,m} = 1$  for all  $r, m \in \mathbb{Z}$ . By (5.32), we have  $b_n = 1$  for all  $n \in \mathbb{Z}$ . Therefore, we can deduce that

$$f_{r,-1}(m, n) = r + m,$$

for all  $r, m \in \mathbb{Z}$ . By Lemma 5.3, we have  $a'_m = 0$  for all  $m \in \mathbb{Z}$ . □

**Lemma 5.5.** *Suppose that*

$$\begin{aligned} Q_{0,s}v_{m,n} &= nv_{m,n+s}, \quad n \neq -s, \\ Q_{0,s}v_{m,-s} &= -s(1 + (s + 1)a'_m)v_{m,0}, \\ P_{r,-1}v_{m,n} &= (b_nr + \lambda + m)v_{m+r,n}. \end{aligned}$$

Then  $a'_m = 0$  for all  $m \in \mathbb{Z}$ . Furthermore,  $b_n = b$  for all  $n \in \mathbb{Z}$ , where  $b = 0$  or  $1$ .

**Proof:** By (4.3), we have

$$nf_{r,-1}(m, n) = nf_{r,-1}(m, n - 1).$$

Then  $f_{r,-1}(m, n) = f_{r,-1}(m, n - 1)$  for  $n \neq 0$ . So we have

$$b_n = \begin{cases} b_0, & n \geq 0, \\ b_{-1}, & n \leq -1. \end{cases}$$

Letting  $h = r, k = -1, r = 0, s = -1$  in (3.8), we have

$$ng_{r,-1}(m, n - 1) = (n - 1)g_{r,-1}(m, n). \quad (5.33)$$

Similar to the proof of Lemma 5.3, assume

$$\begin{aligned} g_{r,-1}(m, n) &= l_{r,m}n, \quad n \geq 0, \\ g_{r,-1}(m, n) &= l'_{r,m}n, \quad n \leq -1, \end{aligned}$$

then we have  $l'_{r,m} = l_{r,m}$  for all  $r, m \in \mathbb{Z}$  and

$$\begin{cases} g_{r,s}(m, n) = nl_{r,m}, & n \neq -s; \\ g_{r,s}(m, -s) = -s(1 + (s + 1)a'_{m+r})l_{r,m}. \end{cases}$$

Similarly, let  $h = 0, k = -1, r = 0, s = 0$  in (3.7), then we can deduce

$$f_{0,0}(m, n) = \lambda + m + \delta_{n,0}f(m).$$

Let  $h = r, k = -1, r = 0, s = 0$  in (3.7), then we have

$$l_{r,m}[\lambda + m + \delta_{n,0}f(m) - \delta_{n,1}f(m+r)] = b_nr + \lambda + m. \quad (5.34)$$

Let  $n = 0$  and  $n = 1$  in (5.34) respectively, then we have

$$l_{r,m}(\lambda + m + f(m)) = b_0r + \lambda + m,$$

$$l_{r,m}(\lambda + m - f(m+r)) = b_0r + \lambda + m.$$

By the fact that  $l_{r,m} \neq 0$  for all  $m, r \in \mathbb{Z}$ , we have  $f(m) = -f(m+r)$  for all  $m, r \in \mathbb{Z}$ . Therefore

$$f(m) = 0, \quad \forall m \in \mathbb{Z},$$

and

$$l_{r,m}(\lambda + m) = b_0r + \lambda + m. \quad (5.35)$$

Let  $n \neq 0, 1$  in (5.34), then we have

$$l_{r,m}(\lambda + m) = b_nr + \lambda + m.$$

It is easy to see that  $b_n = b_0 = b$  for all  $n \in \mathbb{Z}$ . Replace  $r$  by  $-r$ ,  $m$  by  $m+r$  in (5.35) respectively, then we get

$$l_{-r,m+r}(\lambda + m + r) = -br + \lambda + m + r.$$

By (5.20), we have

$$(-br + \lambda + m + r)l_{r,m} = \lambda + m + r. \quad (5.36)$$

Using (5.35) and (5.36), we obtain

$$l_{r,m}(b-1) = b-1, \quad r \neq 0. \quad (5.37)$$

Then  $l_{r,m} = 1$  or  $b = 1$ . If  $l_{r,m} = 1$  for all  $r, m \in \mathbb{Z}$ , then by (5.35), we have  $b = 0$ . If  $b = 1$ , by (5.35) we have  $(\lambda + m)l_{r,m} = r + \lambda + m$ . It is easy to deduce that  $\lambda \notin \mathbb{Z}$  and

$$l_{r,m} = \frac{r + \lambda + m}{\lambda + m} \text{ for all } r, m \in \mathbb{Z}.$$

**Case 1.** If  $l_{r,m} = 1$  and  $b = 0$ , then

$$\begin{cases} g_{r,s}(m, n) = n, & n \neq -s, \\ g_{r,s}(m, -s) = -s(1 + (s+1)a'_{m+r}), \\ f_{r,-1}(m, n) = \lambda + m. \end{cases}$$

By (4.28) we have

$$f_{r,s}(m, n) = rg_{r,s+1}(m, n) + (\lambda + m)g_{r,s}(m, n+1) - (\lambda + m + r)g_{r,s}(m, n). \quad (5.38)$$

Then we can deduce that

$$\begin{aligned} f_{r,s}(m, n) &= \lambda + m, \quad s + n \neq 0, -1; \\ f_{r,s}(m, -s-1) &= \lambda + m - (s+1)(r(s+2) + (\lambda + m)s)a'_{m+r}; \\ f_{r,s}(m, -s) &= \lambda + m + (s+1)(\lambda + m + r)a'_{m+r}. \end{aligned}$$

Let  $n = 2$  in (5.28), then we obtain

$$(\lambda + m)^2 a'_m = 0. \quad (5.39)$$

If  $\lambda \notin \mathbb{Z}$ , then  $a'_m = 0$  for all  $m \in \mathbb{Z}$ . If  $\lambda \in \mathbb{Z}$ , then  $a'_m = 0$  for  $m \neq -\lambda$ . Then we have

$$\begin{cases} f_{r,s}(-\lambda - r, -s - 1) = -r - 2(s + 1)ra'_{-\lambda}, \\ f_{r,s}(m, n) = \lambda + m, \quad \text{for other } m, n, \end{cases}$$

and

$$\begin{aligned} f_{r,s}(m, n) &= \lambda + m - \delta_{m+r, -\lambda} \delta_{s+n, -1} 2r(s + 1)a'_{-\lambda}, \\ g_{r,s}(m, n) &= n - \delta_{m+r, -\lambda} \delta_{s+n, 0} s(s + 1)a'_{-\lambda}. \end{aligned}$$

Letting  $s = 1, k = 0$  in (3.8), we have

$$(1 - 2n)\delta_{m+r+h, -\lambda} \delta_{n, -1} a'_{-\lambda} = 0.$$

Letting  $n \neq -1, m + h + r \neq -\lambda$ , we obtain

$$a'_{-\lambda} = 0.$$

Therefore,  $a'_m = 0$  for all  $m \in \mathbb{Z}$ .

**Case 2.** If  $l_{r,m} = \frac{r + \lambda + m}{\lambda + m}$  and  $b = 1$ , where  $\lambda \notin \mathbb{Z}$ , then

$$\begin{cases} g_{r,s}(m, n) = \frac{r + \lambda + m}{\lambda + m} n, & n \neq -s, \\ g_{r,s}(m, -s) = -\frac{r + \lambda + m}{\lambda + m} s(1 + (s + 1)a'_{m+r}), \\ f_{r,-1}(m, n) = r + \lambda + m. \end{cases}$$

By (5.38), we have

$$\begin{aligned} f_{r,s}(m, n) &= r + \lambda + m, \quad s + n \neq 0, -1; \\ f_{r,s}(m, -s - 1) &= r + \lambda + m - \frac{r + \lambda + m}{\lambda + m} (s + 1)[s(\lambda + m + r) + 2r]a'_{m+r}; \\ f_{r,s}(m, -s) &= r + \lambda + m + (s + 1) \frac{(r + \lambda + m)^2}{\lambda + m} a'_{m+r}. \end{aligned}$$

Letting  $h = 0, k = -2, s = 0, n = 2$  in (3.6), we have

$$(\lambda + m + r)a'_m = 0.$$

Then  $a'_m = 0$  for all  $m \in \mathbb{Z}$  by  $\lambda \notin \mathbb{Z}$ . □

**Lemma 5.6.** *The following case does not exist:*

$$\begin{aligned} Q_{0,s}v_{m,n} &= nv_{m,n+s}, \quad n \neq -s, \\ Q_{0,s}v_{m,-s} &= -s(1 + (s + 1)a'_m)v_{m,0}, \\ P_{r,-1}v_{m,n} &= (b_n r + m)v_{m+r,n}, \quad m \neq 0, m + r \neq 0, \\ P_{r,-1}v_{m,n_0} &= (r + m)v_{m+r,n_0}, \quad m \neq 0, \\ P_{r,-1}v_{0,n_0} &= r(1 + (r + 1)b'_{n_0})v_{r,n_0}, \end{aligned}$$

for some  $n_0 \in \mathbb{Z}$ .

**Proof:** By Lemma 5.5, we have

$$f_{r,-1}(m, n) = \begin{cases} f_{r,-1}(m, 0), & n \geq 0, \\ f_{r,-1}(m, -1) & n \leq -1, \end{cases} \quad (5.40)$$

$$ml_{r,m} = f_{r,-1}(m, n). \quad (5.41)$$

By (5.41), we know  $b_n = b$  for all  $n \in \mathbb{Z}$ . Note that  $b_{n_0} = 1$ , so  $b = 1$  and by the results on representations of the Virasoro algebra, we have

$$\begin{cases} f_{r,-1}(m, n) = r + m, & m \neq 0, \\ f_{r,-1}(0, n) = r(1 + (r + 1)b'_{n_0}). \end{cases}$$

Let  $m = 0$  in (5.41), then we have

$$0 = f_{r,-1}(0, n) = r(1 + (r + 1)b'_{n_0}),$$

a contradiction. □

Similarly we have the following lemmas.

**Lemma 5.7.** *Suppose that*

$$\begin{aligned} Q_{0,s}v_{m,n} &= nv_{m,n+s}, & n \neq -s, \\ Q_{0,s}v_{m,-s} &= -s(1 + (s + 1)a'_m)v_{m,0}, \\ P_{r,-1}v_{m,n} &= (b_nr + m)v_{m+r,n}, & m \neq 0, m + r \neq 0, \\ P_{r,-1}v_{m,n_0} &= mv_{m+r,n_0}, & m \neq -r, \\ P_{r,-1}v_{-r,n_0} &= -r(1 + (r + 1)b'_{n_0})v_{0,n_0}, \end{aligned}$$

for some  $n_0 \in \mathbb{Z}$ . Then  $b'_{n_0} = 0$ ,  $a'_m = 0$ ,  $b_n = 0$  for all  $m, n \in \mathbb{Z}$ .

**Lemma 5.8.** *Suppose that*

$$\begin{aligned} Q_{0,s}v_{m,n} &= (a_ms + n)v_{m,n+s}, & n \neq 0, n \neq -s, \\ Q_{0,s}v_{m_0,n} &= nv_{m_0,n+s}, & n \neq -s, \\ Q_{0,s}v_{m_0,-s} &= -s(1 + (s + 1)a'_{m_0})v_{m_0,0}, \\ P_{r,-1}v_{m,n} &= (b_nr + m)v_{m+r,n}, & m \neq 0, m + r \neq 0, \\ P_{r,-1}v_{m,n_0} &= mv_{m+r,n_0}, & m \neq -r, \\ P_{r,-1}v_{-r,n_0} &= -r(1 + (r + 1)b'_{n_0})v_{0,n_0}, \end{aligned}$$

for some  $m_0, n_0 \in \mathbb{Z}$ . Then  $a'_{m_0} = 0$ ,  $b'_{n_0} = 0$  and  $a_m = 0$ ,  $b_n = 0$  for all  $m, n \in \mathbb{Z}$ .

It follows from Lemmas 5.1-5.8 that there are no other situations except the seven ones in Theorem 3.4.

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DEPARTMENT OF MATHEMATICS, SHANGHAI JIAOTONG UNIVERSITY, SHANGHAI 200240, P.R.CHINA  
 E-mail address: gaoshoulan@sjtu.edu.cn

DEPARTMENT OF MATHEMATICS, SHANGHAI JIAOTONG UNIVERSITY, SHANGHAI 200240, P.R.CHINA  
 E-mail address: cpjiang@sjtu.edu.cn